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ABSTRACT

This is the second module in a series on mathematics methods and materials for preservice elementary teachers. This module focuses on three instructional and learning modes: expository, guided discovery, and inquiry (pure discovery). Objectives for the module are listed, the prerequisites are stated, pre- and post-assessment standards are specified, and 15 instructional activities are suggested. (DT)

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INSTRUCTIONAL AND LEARNING MODES IN MATH

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Objectives

1. (Knowledge): Describe each of the following Instructional and Learning Modes as applied to elementary school math:

   a. Expository Mode
   b. Guided-Discovery Mode
   c. Inquiry (or Pure Discovery) Mode.

2. (Knowledge): Descirbe the manner in which the following Instructional Variables are manipulated for each of the instructional and Learning Modes listed in Objective #1:

   a. Learner involvement
   b. Content structuring
   c. Strategy sequencing
   d. Reinforcement strategies
   e. Plans for motivation

3. (Knowledge): Identify the significance of the following additional Instructional Variables for each of the Instructional and Learning Modes listed in Objective #1:

   a. Time
   b. Retention
   c. Resources
   d. Teacher role

4. (Performance): Demonstrate his/her ability to describe an observed teacher's utilization of the Instructional Variables listed in Objectives #2 and #3 and to identify to which Instructional and Learning Modes the descriptions belong. (NOTE: A copy of the "Observation Guide: Instructional Variables Within Instructional and Learning Modes" is attached.)

5. (Performance): Demonstrate the application of the essential elements of each Instructional and Learning Mode with a small group of learners; satisfactory completion to be evidenced by the inclusion of at least seven (7) of the essential elements of each mode. (NOTE: There are three attached sheets, Application of Instructional and Learning Modes to be used with this objective.)

6. (Product or Consequence): Gather data describing behaviors of learners during at least one of the application experiences completed in Objective #5. (NOTE: See the attached sheet, "Gathering Data Describing Learners Behaviors.)

Pre-Requisites

Module CMM:006:01 (ISU) "Orientation to Math Methods and Materials."

SE 040 620

Pre-Assessments

Objectives 1, 2 and 3---Knowledge Objectives:
The Intern will complete the Pre-Assessment with at least 90% consistency with the Keyed Response.

Objectives 4 and 5:---Performance Objectives: Objective 6--Product or Consequence Objective:
The Intern cannot complete Pre-Assessments for these objectives since they require application experiences.

Instructional Activities

(NOTE: The Intern may choose from among the following Instructional Activities or design and carry out your own Instructional Activities related to the Objectives of the Module. All Instructional Activities are optional except the required application Instructional Activities needed to satisfy Performance Objectives and Consequence Objectives.)

1. Read attached handouts:
   a. "Modes of Instruction and Learning in Math."
   b. Hendrix: "Learning by Discovery."
   c. Kersh: "Learning by Discovery: What is Learning?"
   d. Kersh: "Learning by Discovery: Instructional Strategies."
   e. Arnold: "Flow Charting the Components of Teaching Strategy."
   f. Lippitt: "The Neglected Receiver: The Child."
   g. O'Daffer: "On Improving One's Ability to Help Children Learn Mathematics."
   h. Torrance: "Characteristics of Learning Experiences to Facilitate Creative Behavior."
   i. "Games Children Play....to Learn."
   j. Davis: "Discovery in the Teaching of Math."
   k. Junge: "Depth Learning in Arithmetic: What is It?"
   l. Lavatelli: "Early Education for the Thinking Child---A Piaget Program in Action."
   m. Kohl: "On Finding Private Spaces."

2. Observe elementary math instruction at any level. Selectively determine what aspects of Instructional and Learning Modes you wish to observe. If you observe in pairs (2 Interns) or conference with the Cooperating Teacher before and/or after the instruction, it might assist your focusing upon specific elements. Avoid evaluating the observed instruction and learning.

3. Participate in a small group or team discussion of the three Instructional and Learning Modes and design, as a group or individually, an example of an instructional sequence representing an application of selected elements of all three of the Instructional and Learning Modes. (NOTE: You may find the handout included with Module CMM:006:01 (ISU) "Facilitating Inquiry in the Classroom.")

4. Refer to handout by Kersh; "Learning by Discovery: What is Learned" and try out the two tasks described (p. 3)--"Word Task" and "Arithmetic Task."

Instructional Activities (continued).....

5. Refer to handout by Kersh: "Learning by Discovery: Instructional Strategies," and try out the two discovery methods described--"Providing Answer-Giving Instructions and Withholding Answer-Seeking Instructions" and "Giving Answer-Seeking Instructions and Withholding Answer-Giving Instructions."

6. Refer to handout by Arnold: "Flow Charting the Components of Teaching Strategy" and identify the Exposition Moves (pp. 1-2), Illustration Moves, (p. 2), Demonstration Moves (p. 2), Discussion Moves (p. 2), Exploration Moves (p. 2), Class Management Moves. Then develop and teach a Flow-Chart Lesson plan. Emphasize one of the Instructional and Learning Modes identified in this Module.

7. Refer to handout by Lippitt: "The Neglected Receiver: The Child" and discuss with another Intern, Cooperating Teacher or Team Leader the "conditions of an effective Learning Situation."

8. Refer to handout by O'Daffer: "On Improving One's Ability to Help Children Learn Math". (a) analyze a bit of instruction using the "tag the things you teach" procedure (pp. 2-4).
   (b) Design a set of teaching procedures using the outline described on pp. 5-7 and try it out with a group of learners.
   (c) Gather some information of how much you recite during a short lesson with a group--tape record the session. Then, try to reduce your amount of talk with the same group at a different time--again, tape record the session.
   (d) Identify your questions used during some sequence with children. Then compare your questions with those suggested on pp. 8-10.

9. Refer to the handout by Torrance: "Characteristics of Learning Experiences to Facilitate Creative Behavior," and try out activities new to you from each list--before the lesson, during the lesson, after the lesson. You may also wish to discuss items in this handout with which you strongly agree or disagree with other Interns, Cooperating Teachers or Team Leader. Some items may be particularly stimulating---Before: 5, 6, 11, 13; During: 7, 10, 12, 16; After: 1, 13, 18, 19, 24, 29, 33.

10. Refer to handout, "Games Children Play... To Learn" and identify the characteristics of a good game described in the handout; then apply the characteristics as criteria to some classroom math games to see how they hold up. Modify the games to suit the criteria of a good game and try them out with learners.

11. Refer to Junge's handout: "Depth Learning in Arithmetic--What is It?" and try out some of the suggested exercises listed. Remember: These exercises stress thinking and involving of learners and do not require teacher (or Intern) explanations to any great dégree.

12. Refer to Kohl's handout: "On Finding Private Spaces" and then interview several learners about how they feel about the need for certain things best done in private. Try to determine how they perceive how the school views their needs for such privacy.

13. Refer to Lavalelli's handout: "Early Education for the Thinking Child--A Piaget Program in Action" and try out some of the many suggested orientation activities or games.

14. Refer to Davis' handout: "Discovery in Teaching of Math" and contrast his views of discovery with some other handout in this Module or contrast his views with those held by a Cooperating Teacher, another Intern, yourself, or your Team Leader.

15. Design and carry out your own Instructional Activities related to the objectives of this Module. Remember: The activity should be designed to help you to be able to describe Instructional and Learning Modes in Math, Instructional Variables to be manipulated, or applications of these competencies.

Post-Assessments

1. Successful completion of Performance Objectives (#4). Submit the completed Observation Guide: Instructional Variables Within Instructional and Learning Modes.

2. Successful completion of Performance Objective (#5). Submit three completed "Application of Instructional and Learning Modes" sheets. Remember, they must demonstrate at least 7 of the essential elements of EACH mode!

3. Successful completion of Porduct or Consequence Objective (#6). Submit a completed report of "Data Gathered Describing Learner's Behavior." This report should include at least samplings of raw data (actual, verbatum) of learners.

4. Completion of Post-Assessment with at least 90% agreement with the Keyed Responses. The Pre- and Post-Assessments should be attached to the Module Record Form.

Remediation

It may be necessary to complete some of the Instructinnal Activities not attempted prior to attempting the Post-Assessment if the Intern does not satisfy the minimal performance on Knowledge or Performance Objectives. If these Objectives prove to be extremely difficult, you may wish to confer with the Team Leader and , if necessary, request a seminar meeting with the Instructor.

MODULE FIELD-TEST EVALUATION FORM

INTERN

MODULE # _____ TITLE _____

COURSE # _____ MODULE BEING STUDIED BY _____ (INTERN)

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1. INDICATE WHICH CATEGORY YOU BELIEVE BEST DESCRIBES YOUR EVALUATION OF THIS MODULE: (CHECK ONE)

____A. KEEP-AS-IS
____B. DELETE ENTIRELY
____C, MODIFY BY DELETING PORTIONS (UNDER #2 IDENTIFY WHICH PORTIONS SHOULD BE DELETED.)
____D. MODIFY BY ADDING PORTIONS (UNDER #2 IDENTIFY WHICH PORTIONS SHOULD BE ADDED.)
____E. MODIFY BY RE-WRITING OR RE-DESIGNING PORTIONS (UNDER #3 IDENTIFY THE KINDS OF RE-WRITE OR RE-DESIGNING YOU VIEW NEEDED--BE SPECIFIC.)
____F. OTHER (SPECIFY UNDER #2).

2. SUGGESTIONS FOR MODIFICATIONS OF MODULES: (FROM #1--C, D, E, OR F ABOVE).

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3. HOW DOES THIS MODULE SATISFY THE FOUR CRITERIA FOR A MODULE LISTED BELOW? (YOUR COMMENTS ARE WELCOMED).

	NOT AT ALL	LITTLE	SOME	VERY WELL
A. "FIELD-ORIENTED"--(UTILIZES KNOWLEDGES AND SKILLS IN APPLICATIONS WITH LEARNERS IN THEIR USUAL LEARNING ENVIRONMENTS IN THE SCHOOL AND COMMUNITY):	_____	_____	____	_____
COMMENTS:				
B. "COMPETENCY-BASED"--(REQUIRE DEMONSTRATION BY INTERN OF CRITERION-REFERENCED COMPETENCIES IN KNOWLEDGES, PERFORMANCES, SKILLS, ATTITUDES, ETC.)	_____	_____	____	_____
COMMENTS:				

MODULE FIELD-TEST EVALUATION FORM

INTERN

	NOT AT ALL	LITTLE	SOME	VERY WELL
C. "PERSONALIZING"--(ALLOWS INTERN TO CHOOSE FROM AMONG ALTERNATIVE INSTRUCTIONAL ACTIVITIES INCLUDING DESIGNING HIS/HER OWN, RELATED TO THE OBJECTIVES OR COMPETENCIES OF THE MODULE):				

COMMENTS

	NOT AT ALL	LITTLE	SOME	VERY WELL
D. "EMPHASIZE SATISFACTION OF PRODUCT OR CONSEQUENCE OBJECTIVES"--(EMPHASIS UPON LEARNER PERFORMANCE OUTCOMES--WHAT CHANGES TAKE PLACE IN CHILDREN AS A CONSEQUENCE OF THE INTERN'S INSTRUCTIONAL EFFORTS):				

COMMENTS

TC OFFICE

DATE EVALUATED: _____

EVALUATORS NAME _____

CHECK ONE:
- INTERN
- TEAM LEADER
- COOPERATING TEACHER
- INSERVICE TEACHER
- OTHER _____ (IDENTIFY)

MAIL OR SEND TO TC OFFICE I.S.U., POCATELLO

MODULE RECORD FORM

Intern _____

Module No. _____ Title: _____

Date Started: _____

Date Completed: _____

Record of Team Leader Conferences: (Team Leader please initial)

Dates	TL Initials
_____	_____
_____	_____
_____	_____
_____	_____
_____	_____

Record of Instructor Seminars: (Instructor please initial)

Dates	Inst. Initials
_____	_____
_____	_____
_____	_____
_____	_____
_____	_____

Written materials completed. (Please attach Pre- and Post-Assessments and written records on Instructional Activities.)

Send this form and attached materials to the Teacher Corps Office.

For TC Office use ONLY:

Included in _____ course credit.

Date Module Satisfied _____ Approval _____

REMARKS:

"MODES OF INSTRUCTION AND LEARNING IN ELEMENTARY SCHOOL MATH"

OVERVIEW OF INSTRUCTIONAL MODES

Inquiry or Pure Discovery: When a learner formulates his own questions or hypotheses, develops his own strategies for answering the questions or testing the hypotheses, collects data and answers his own questions, he is engaging in "pure discovery or inquiry." Pre-schoolers engage in pure discovery or inquiry extensively. As they explore the environment, natural curiosity and wonder lead them in the discovery or inquiry process.

Guided Discovery: When a problem or question is posed "externally" and the learner develops strategies, collects data and answers the questions with teacher help as needed, he is engaging in "guided or directed discovery." When a child is presented with a set of data, formulates hypotheses and tests them with teacher guidance as needed, he is engaged in guided or directed discovery. Problems are posed and materials for their solution are made readily accessible, or a set of data is provided and the child is helped to "make sense of" or extract a generalization from the data.

Expository: When a problem is posed and answered "externally" and the child is expected to replicate the problem and solution or to work others which are similar, he is engaging in "expository instruction." Lectures and explanations are expositorily presented. The learner is likened to a receptacle which is being filled. Nearly all secondary school and college instruction and much elementary school instruction is expository.

PURE DISCOVERY OR INQUIRY MODE

Scientists and mathematicians hypothesize relationships and then seek to prove or disprove them. Adults encounter impasses in their personal lives concerning relationships with people or with technology. They define problems, think of possible alternatives, and act. No one knows for sure how people learn to define and solve problems, but there is surely some value in practicing problem-solving behaviors. Pure discovery or inquiry provides children with such opportunities. Examples:

1. Children are encountering Cuisenaire Rods for the first time. They are given rods for independent exploration. What are some things they "might" find out (discover) without teacher guidance?
   a. Cuisenaire Rods make tall towers.
   b. The orange rod is longer than the dark green rod.
   c. Red and blue placed end-to-end have the same length as blue and red placed end-to-end.
   d. The rods can be grouped into color families.
   e. The rods can be placed in a stair-step pattern, etc....

2. Third graders are given several containers of standard size, i.e., cup, pint, quart, half-gallon. They take them to the sandbox to play. What might they find out (discover) without teacher involvement?
   a. Some jars hold more sand than others.
   b. Some jars hold multiples of other jars.
   c. Two cup containers fill a pint container.
   d. A container cannot be "exactly" filled.
   e. A more precise or descriptive measure can be obtained by using smaller and smaller units, etc....

3. Fifth graders are given many solid objects and asked if they can find any relationships among the objects. What are some facts they might discover without further teacher involvement than setting the problem?
   a. Some objects are smooth; others are rough.
   b. Some objects have flat portions; others do not.
   c. Some objects are slanted; some are not.
   d. Some objects roll; others do not.
   e. Some objects have vertices; others do not, etc.....

Characteristics: "Might" is emphasized in each of the preceding illustrations. Pure discovery or inquiry learning experiences are very unstructured and open-ended. The child poses questions, selects a route towards resolution of the question, and decides on the validity or merit of his results. The learning is very internal and self-centered.

The learner asks questions which are meaningful to him, uses strategies for resolution which are within his experience and grasp, and determines his own success. The learning is very intimate and personal; it has meaning and significance. There is no tangent; the child is right on target because he is allowed to seek his own meaning. The questions raised are the questions for which answers are desired.

If the teacher expects the child to discover something in particular, the experience is not pure discovery or inquiry experience. Pure discovery or inquiry is characterized by randomness, trial and error and absence of externally imposed structure.

Objectives Attained: This mode is excellent when you wish to provide learners with unstructured opportunities to formulate and evaluate hypotheses. Pure discovery or inquiry, because of its inherent freedom, permits much opportunity for creative expression by providing stimulating opportunities to explore, discover, compare and evaluate. It is a good motivational experience since children can explore mathematical teaching aids for enjoyment without the pressure of "having to learn something."

Appropriate Uses: The objectives of the previous paragraph can be attained through several occasions which occur throughout the K-6 curriculum:

1. When new teaching materials are introduced, a free-play period or time of exploration can be provided. This has the dual impact of acquainting learners with the materials while providing opportunities for new discoveries.

2. Children can be given access to mathematics-related materials during recess and other play periods.

3. After a structured presentation, pupil exploration may yield teacher insights relative to pupil transfer of ideas to new materials.

4. Children can be given free play exploration time prior to a more structured sequence of guided discovery or expository instruction.

Appraisal: In relation to other instructional criteria, the following statements can be made about pure discovery or inquiry:

1. Pure discovery or inquiry is a good motivator for most students. When the learner is given opportunities to operate in an unstructured setting in which he can "go his own way," he is motivated to follow his own interests. He relates well to his own quest behavior and is involved in learning experiences. The appeal to the child's curiosity is a natural motivator.

2. Pure discovery or inquiry does not enhance the learner's knowledge of structure except in a very coincidental or random way. If the learner discovers structure, that is well and good, but the nature of the mode is such that there is no assurance this will happen. The experience can be structured to emphasize the concrete as a starting point, but the child can pursue as he sees fit.

3. Sequence is similarly coincidental. The idea of sequence is rather antithetical to the unstructuredness of a pure discovery or inquiry learning experience, except that it can begin with the concrete. The sequence of learning activities is determined by the learner rather than the teacher.

4. Since the child asks his own questions and answers them, success is almost assured. Success is determined internally. The child receives immediate feedback which is almost certainly understandable, and he is relatively sure to answer his questions which means success is almost a certainty.

In summary, pure discovery has a place in elementary school mathematics instruction. It rates highly on motivation and success and contributes to the attainment of broad objectives relative to strategy formulation, hypothesizing, and data gathering.

GUIDED DISCOVERY MODE

Most instruction which carries the label "discovery" is some form of inductive learning, i.e., the child determines a generalization and the instances to which it applies. Children are discovering when they use problem-solving behavior. The steps can be roughly described as follows:

1. Awareness--The learner becomes aware of the problem. He interprets, rephrases and redefines it.
2. Strategy formulation--The learner formulates a strategy.

3. Data collection--The learner gathers information and reflects on relationships.
4. Hypothesis formulation--The learner hypothesizes relationships.
5. Hypothesis testing--The learner tests hypotheses and accepts or rejects them.

When directed discovery strategies are used in elementary school mathematics, the teacher assumes the role of facilitator. The learner's progress can be facilitated by the teacher, particularly in steps 2 and 4 above. The teacher is working with the learner in step 2 to facilitate formulation of a workable strategy. These two guiding or facilitating processes have different objectives which deserve special consideration.

Facilitating Strategy Formulation: When teachers engage in this behavior, they are helping learners learn how to solve problems. When teachers pose questions which have several acceptable strategies, they help children learn that a problem can be attacked in more than one way. This procedure helps assure the teacher that the learner is attacking the problem in a way which has meaning. There are many ways to solve problems, but some strategies are more sophisticated or elegant or efficient than others. When a teacher exposites, all the children learn by one strategy, the teacher's strategy. Helping children evolve their own strategies helps ensure that the procedures being used are meaningful to the learner, and the learner acquires experience and skill in the more general task of problem-solving.

Facilitating Hypothesis Formulation: When teachers engage in this behavior, they are facilitating pupil discovery of the particular genralization being examined. Use of this approach helps children develop an understanding of the generalization by extracting it from a set of data. This approach can be used as one of the steps in the five step problem-solving process.

When the five step process is used, several simple story problems can be distributed to small groups. Each group works the problems and the teacher helps them set up the various number sentences and offers hints to foster recognition of the generalization.

When awareness, strategy formulation and data collection (1, 2, and 3) are omitted, the teacher can explore these steps with the class before working in groups. The teacher can establish a strategy, set up number sentences and ask the children to look for the generalization. The problem-solving strategies are presented expositorily and the learners discover the generalization under study.

Another example of this approach is in the use of conjured mathematics examples to help learners experience the enjoyment and pleasure of making discoveries. Activities of this sort are found in materials of the Madison Project and in most current textbook series. Robert Davis has children look for numbers which can be used in place-holders of linear and quadratic equations or discover generalizations with geoboards. Example:

Name some number pairs of the "truth set" for $\triangle + \square = 14$.

Characteristics: In guided discovery, children are involved. Challenging questions are raised to stimulate interest and curiosity, and appropriate materials are provided with which children can discover solutions. The teacher asks the questions, so guided discovery has structure. However, children are given freedom to answer the questions in ways which are consonant with their abilities and levels of understanding.

Children seek out personalized solutions; most of the motivation is internalized. Since the teacher guides and directs children's efforts, success is not guaranteed but it is greatly enhanced. Nearly all children will experience success. The teacher can give relatively more guidance to those who are slow or who are having difficulty identifying an appropriate strategy or generalization.

Objectives Attained and Appropriate Uses: This mode is excellent when you wish to provide learners with opportunities to develop problem-solving skills and meaningfully to learn new generalizations from analysis of patterns in data. It is self-motivating as a strategy so it enhances interest and fosters good attitudes towards mathematics. Guided or directed discovery can be used advantageously when new concepts are being introduced and when generalizations are to be learned. Its intent is to pose questions which challenge learners to formulate problem-solving strategies, collect data and verify or reject hypotheses. Small group, laboratory type settings are ideal for such activities because they offer rich opportunities for peer group interaction and manipulation of concrete materials in the discovery process.

Appraisal: In relation to the four instructional criteria mentioned in the first section, the following statements can be made:

1. One of the clearest strengths of discovery strategies is the strong motivational effect. The heightened interest of students who are involved in this mode seems to increase practice and learning. After the teacher utilizes an appropriate arousal strategy, the activities become more or less self-sustaining. The teacher aids learners who have arrived at an impasse and curiosity maintains interest.

2. Since the teacher controls the presentation of concepts to be discovered, this mode can be used very effectively to present mathematics in a way which is structurally clear. Studies by several researchers have supported the hypothesis that discovery modes facilitate transfer of both generalizations and problem-solving strategies (heuristics).

3. Children can work with materials appropriate to their levels of understanding, so sequencing of learning activities can be controlled. The teacher can allow considerable latitude in learner strategies; children can proceed from concrete to abstract as quickly or as slowly as they deem necessary and expedient.

4. Success is determined externally in most cases but the child is given help through hints relative to strategies of data analysis which help assure success. The child is given whatever help he needs to make the discovery.

4. (continued).....
The reinforcement and help will foster better attitudes.

Many guided discovery strategies will be made available to Trainees during module study.

EXPOSITORY INSTRUCTION MODE

Teachers are notorious for their talkativeness. They probably have earned this distinction through the teaching style which typifies most instruction. Children and adults obviously learn when instructed in this style since most people have experienced such instruction in much of their formal education. Some of the characteristics, uses, and advantages of expository instruction are explored in this section.

Characteristics: Much expository instruction is "explaining." Teachers tell and show learners how to verify mathematical properties, how to use concepts, and how to compute. Explaining often takes the form of a "deductive sequence," i.e., the teacher presents a generalization or concept and gives several examples to demonstrate the uses or attributes. Following explanations or deductive sequences, students practice procedures or identify additional examples.

If the teacher expositorily presents the concept of commutativity of multiplication, $a \times b = b \times a$, the concept may be "demonstrated" on a number line, with sets and with learning aids, and then as number sentences. The generalization is given and explained; then the students practice for understanding and retention.

Expository instruction is a rather "closed" sequence. The teacher decides what will be studied (asks the questions), teaches it (answers the questions), and evaluates learner practice on the concepts (determines success). There are few opportunities for learner divergent thinking. The teacher who utilizes this mode must, because of its closed nature, assume most of the responsibility for sustaining motivation and interest through use of "good" materials and illustrations.

Expository instruction is also characterized by the narrow range of pupil skills allowed at any particular time and the lock-step pace which it requires. Learners practice the skills or generalizations as they are presented and each student must move ahead as the presentation moves from one new topic to another. If the class is divided into two or three groups, this weakness is not as prominent but is still to be reckoned with.

Objectives Attained Through Appropriate Uses: Expository instruction allows you to present a large amount of material in relatively short periods of time. It is particularly useful when you need to teach ideas which are not easily discovered or to review previous work. (NOTE: "Presenting of material or concepts" should not be assumed to be the same material or concepts which are "learned or internalized." This is often a prime fallacy of expository instruction!)

Learners cannot "discover" vocabulary or common symbols or notations. They must be told the names of concepts which have been learned and they need to be

told how to write an equation or the standard notation or the way to set up a problem in vertical form or the symbols which represent addition, equal, place holder, etc.

After a concept or generalization has been discovered, expository instruction is appropriately used to review the concept, to point out its attributes and to explore its ramifications. If the students have discovered the essential ideas of regrouping in addition, you can help them write a computational procedure which describes symbolically those operations performed with concrete materials, and you can work some examples with them.

Appraisal: As you know, expository instruction can be and often is immensely boring. The skill, interest and enthusiasm of the teacher is a critical factor in its success or failure. As a method of instruction, the success of expository instruction hinges on the teacher. The strategy itself is not particularly motivating since the child is perceived as a receptacle to be filled. It is not very exciting and tends to arouse little natural curiosity. Since there is little challenge, arousal strategies are needed, and external incentives (rewards) must usually be employed. Broad objectives of internalized incentives and positive attitudes are difficult to attain in this mode.

Mathematical structure can be taught in this mode with the major drawback being that most learners must progress at about the same rate. Some will move through the structure too slowly and others too quickly. Transfer can be built into such a manner or presentation since several studies have indicated that meaningful instruction, instruction which emphasizes generalizations and instruction which emphasizes relationships between concepts and transfer-like situations, all enhance the amount of transfer likely to occur.

The teacher can control the sequence, too, but the same weakness of lock-step instruction will tend to decrease effectiveness. Children will not be ready to follow the sequence simultaneously.

Since the teacher asks the questions, success is determined externally. Some learners will experience success nearly constantly, but some will experience success rarely since the structure and sequence of a lock-step curriculum is inappropriate for their needs.

SUMMARY AND GUIDELINES FOR THE THREE MODES

Below the three modes are collectively related to the criteria established. Other important variables are briefly discussed and summarized comments are presented.

Three Modes: A summary of the three modes is presented in the following table relative to the four criteria used. As a general mode of instruction, expository is weakest on the basis of these four criteria and guided discovery is strongest.

	Motivation	Structure	Sequence	Reinforcement
Pure Discovery or Inquiry	Good	Weak	Weak	Good
Guided Discovery	Good	Good	Good	Good
Expository	Weak	Fair	Fair	Fair

Discovery strategies tend to be better motivators and reinforcers than expository strategies because the learner is involved in decision-making and evaluating his progress. He chooses his own questions or designs his own strategies and receives teacher guidance or decides his own success. Expository strategies tend to rely more upon external incentives and success criteria. The broader society desires education for its youth which instills love of learning. Discovery modes seem to contribute to this goal; they offer opportunities for development of problem-solving skills and divergent thinking. They make the child feel more positive about his abilities and, hence, himself.

Guided discovery does the most adequate job of presenting material in a sequence which is appropriate and which helps the learner grasp the structure of mathematics. It facilitates greater transfer. Guided discovery strategies, especially if used in small group work, offer opportunities for learners to be working at all three levels of abstraction in the classroom at one time. The teacher as facilitator helps learners move meaningfully from one level to another.

This discussion of modes has NOT been to convince you that you should use one mode exclusively. You should be aware of the strengths, weaknesses and appropriate uses of each mode. Your instruction should include special consideration of the mode or modes which will work best for you in helping you attain your objectives, to the extent that your objectives coincide with the four utilized in this handout.

Other Important Variables: The four criteria utilized are certainly important in building a solid curriculum. There are other variables which also influence day-to-day decisions. Some of these will now be considered. These may give you additional insights and help you make intelligent decisions.

Time: The time devoted to presentation in expository instruction is generally shorter. Discovery modes may require more time because of the amount of "facilitation" needed. You should remember, however, that transfer is greater in discovery modes so the extra time may be well spent.

Retention: Instruction in any mode can be meaningful instruction but the likelihood that it will be meaningful for the larger number of students favors a guided discovery mode since children are more inclined to be operating at an appropriate level of abstraction. Retention is greatest when 50 to 75 percent of instructional time is spent in developmentally meaningful activities and the remainder (less than half) of the time on practice or drill. Worthen found that while expository instruct-

ion yielded the greatest initial learning, discovery learning yielded greatest learning after five and eleven weeks.

Resources: Availability of appropriate aids might seem to be an important factor, but you will see in succeeding chapters that the resourceful teacher is surrounded with potential mathematics teaching devices. Rocks, tin cans and magazines are representative of items which are potentially useful in small groups or individual work.

Teacher Role: Your perception of your role in the classroom is a fundamentally important variable. If you are rigidly authoritarian, you will have difficulty with discovery modes. If you are bothered by student boredom or listlessness, you probably will seldom use expository instruction for concept presentation. The three modes run a gamut of structure and teacher control of pupil learning. Hopefully, you can use all three modes to accomplish the objectives for which each is best suited.

SUMMARY

The mode of instruction is an important variable in the teaching-learning process; so is the teacher. Success or failure of instruction is highly dependent upon the teacher's ability to carry it off. Ausubel points out that expository instruction can be meaningful instruction. Much expository instruction is not meaningful, but that is the fault of the teacher and not the mode. He also points out that guided discovery strategies can be rote learning experiences. Teachers in the role of facilitator assume a responsibility which requires some skill. Most conscientious teachers can probably develop the skill.

Mathematics instruction is much more than skill development. Children need enjoyable experiences which foster positive attitudes and instill natural curiosity about learning in general and mathematics in particular. The best instructional decision is one which reflects a wise fit of modes to achieve short-range and long-range objectives.

LEARNING BY DISCOVERY

Gertruce Hendrix*

An analysis of some methods of teaching, considering the question "What is the discovery method of teaching?"

The issue of learning by discovery and of teaching for learning by discovery is beclouded at present by the fact that each of three very different procedures is being called 'The Discovery Method.' They are the inductive method, the nonverbal awareness method, and the incidental method.

THE INDUCTIVE METHOD

Many people think that 'the discovery method' is just another name for the inductive method. The inductive method is nothing new in mathematics education. Colburn's book on teaching arithmetic by this kind of approach was first published in the early 1820's. The The companion volume on algebra came a few years later. This method was strongly advocated in reports of the National Committee on Reorganization of Secondary School Mathematics in 1918 and 1923. That committee of the Mathematical Association of America drew heavily on earlier recommendations of E. H. Moore in the United States, Perry in England, and Felix Klein in Germany--mathematicians who were writing of these things around the turn of the century. Again in the early 1940's the Joint Committee of the National Council of Teachers of Mathematics and the Mathematical Association of America, in a report published as a yearbook of the National Council, strongly advocated the inductive method. These recommendations, however, were acted upon by such a small minority of teachers of mathematics that most people seeing the inductive method in action today think they are seeing something new. Furthermore, in clarity and productive outcome it is so far superior to an authoritarian "tell'm and drill" approach that, to someone whose school experience has been dominated by a formal approach, the inductive method seems the final answer to all pedagogical difficulties. Hence, someone just new beginning to think seriously about theory of instruction is likely to look upon the inductive method as the ultimate answer. The fallacy in the inductive method lies in its confusion of verbalization of discovery with the advent of the discovery itself.

The separation of discovery phenomena from the process of composing sentences which express those discoveries is the big new breakthrough in pedagogical theory. In some cases, correct verbalization of a discovery does emerge immediately after the dawning of awareness, but these cases are rare. They are confined to learners with unusual powers of correctness and precision in the use of a mother tongue, and they are confined to situations in which the learner happens to possess all the vocabulary and rules of sentence structure needed to formulate the new discovery. In most cases, for evidence that an inductive discovery has occured, a teacher must be able to recognize responses or behavior other than a linguistic formulation of the discovery. The widespread confusion of sentence formation with the discovery process itself has been responsible for much of the frustration, discouragement, and ultimate abandonment of the inductive method on the part of many teachers who have attempted to master it. As one head of a teachers college department of education has put it, "But the only ones who can state a generalization from examining examples are the same exceptionally bright ones who could have learned to apply the generalization from studying someone else's statement of it in the first place!" As a matter of fact, the only teachers who have been really successful with the inductive method are the rare ones who have acquired the knack of making the learner aware of generalizations before calling for verbalizations. They have been doing "right things for wrong reasons," and the subsequent untimely verbaliza-

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tions have only partially marred the outcomes.

If the writer may be permitted so personal a statement, I should like to record here that it was while in pursuit of greater skill with the inductive method that I first became conscious of the nonverbal awareness stage in discovery learning. From concentrated work in helping student teachers to acquire skill in this method, I had become very conscious of three bad mistakes which beginners are likely to make when they plan and attempt inductive teaching:

1. They begin to call for generalizations, exerting anxious pressure on their classes, long before the students have noticed any basic similarity among the examples the teacher has presented. The discussions degenerates into a guessing contest during which the students try to find out "what it is the teacher wants me to say."

2. The teacher often calls for statements of generalizations when the students do not possess the vocabulary and rules of sentence formation necessary for precise verbalization of the generalization, even if they have "seen it."

3. The teacher often confuses generalizations, which had to be discovered in the first place and, hence, are appropriate subject matter for rediscovery, with situations in which the generalization is arbitrary--something that is merely a matter of definitions.

A frequent costly consequence of item 2 above is a teacher's yielding to a temptation to accept a faulty answer, an answer which is either false or nonsense. Spontaneous verbal responses at moments of discovery are usually expressive of emotion only. Often they are ejaculations, referentially haphazard. Sometimes they are in sentence form but seem to be utter nonsense if repeated and examined. Or, still worse, they may be plausible, interprecable sentences which state false generalizations. Spontaneous verbal fragments often provide clues from which a clever teacher can infer what is in a student's mind, and for that purpose they are valuable. But, unless the teacher is a tutor working with only one pupil, encouraging such responses can be very damaging to the learning under way in other members of the class. Even in the tutoring situation, an incorrect spontaneous verbalization of a discovery--a sentence which really says something else--must be ignored and forgotten as soon as possible. Accepting it as a "nice try" can cause an involuntary interpretation of the false sentence to prevail over the correct discovery. If any recognition is accorded to such a response, the teacher must stick with it until counter examples have revealed to the whole class that the sentence must be discarded. By this time the class has shifted to an exercise in language composition, which cannot be abandoned tactfully. Pushing on through to a correct verbalization at such a time usually demands a long, laborious digression. By the time the point is clear, most of the zest for the original learning has evaporated; and the production of the correct sentence, accepted at long last, may even destroy the anticipatory attitude with which ensuing applications of the generalization should be approached. (Experimentation has established that this effect of finality and detachment is avoided if verbalization of a generalization is postponed to a later lesson. At that time the linguistic formulation of things already "known" can be undertaken as an end in itself.)

The big problem in the inductive method has been to find some way of telling when students are far enough along in an inductive approach for it to be fair to ask them to state a generalization. It was while a search was under way for a criterion by which to identify this stage in each inductive lesson than an event came along which led to my identification of the nonverbal awareness stage in discoveries of concepts and generalizations. This happened in 1937.

As early as 1946, clinical experts in nondirective psychological counseling were pointing out that frequently there were very significant changes in behavior based on unverbalized insights, and that the effectiveness of an interview could be destroyed by premature pressure for verbalization. The counseling experts were not telling us, however, how to detect when it was not too soon to press for verbalizations. When I designed my first experiment for research on this question for one group of subjects the teaching was terminated at the nonverbal awareness stage simply to call attention to the fact that that stage existed, that there was a way to tell when a learner was aware of a generalization which he might be asked to state. At the time that I designed this part of the experiment, it did not occur to me that as far as transfer power was concerned, the whole thing was there as soon as the nonverbal awareness had dawned. No one was more shocked than I was to find that not only did the learners who completed correct verbalization of the discovery do no better on transfer tests than those for whom the teaching was terminated at the nonverbal awareness stage, but also that the verbalization of the discovery seemed actually to have diminished the power of some persons to apply the generalization. Charles Hubbard Judd in his conscious generalization theory of transfer had considered 'generalizing' as synonymous with 'composing a sentence which states the generalization involved'. But that simply is not the case; unless Judd's definition of 'generalizing' is changed, we can no longer say that generalizing is the primary generator of transfer power.

Since 1946, the problems have become:

a) How can teaching be planned and executed so that necessary verbalization of discoveries are accomplished without damage to the dynamic quality of the learning itself?

b) What is there about the nature of language and the process of acquiring a language, and what goes on when one uses language that could explain (and, hence, help us to avoid) detrimental effects of verbalization?

Very soon a tremendously important by-product of this search began to appear: The discovery process itself is so exhilarating (to both children and adults) that it becomes its own motive in academic work. As a solution to the motivation problem for children in a free society whose existence is threatened by rapidly increasing scientific and technical power in the land of an enemy, the promotion of successful learning by discovery has no parallel. But such teaching is an art with a difficult technique. Furthermore, it is dependent upon sound and clearly written textbook materials. Ambiguities, inconsistencies, "fuzzy" writing tolerated for centuries--all these must be climinated from science and mathematics courses. (Another field in which such stumbling blocks abound is that of grammatical construction of word languages. The unifying concepts common to all languages are now revealed by developments of the last half century in symbolic logic, but as yet these developments have not been utilized in a reorganization of grammar.)

THE NONVERBAL AWARENESS METHOD

This method has been described incidentally in the above discussion of the inductive method. It will be further explained, perhaps, in the illustration developed in the second part of this article. This is the method of teaching which has been found most successful with the UICSM Mathematics Project materials. The materials are written to promote use of the method with a minimum of difficulty for a teacher to whom both materials and method are new.

THE INCIDENTAL METHOD

This is an approach widely promoted during the Progressive Education era. It is the method in which the school sets the stage for many experiences, which are usually being

built around some central problem or project. Those who advocated this method knew that only generalizations which emerged from experiences were likely to play a dynamic part in the learner's later behavior and problem-solving activity. But all too often they took no responsibility for seeing that instances of the same generalization came along close enough together for the learner to become aware of either concepts or principles. With no underlying explanatory theory, those educators simply did not realize that most learners under such conditions seldom "broke through" to awareness of abstractions. Such a school program was doomed to triviality, except in the hands of a teacher who did many additional things, even though he did them for unsound reasons. Many persons today associate this triviality of results, from any of the so-called 'activity programs' with learning by discovery. They thus reject anything called 'discovery method'.

ILLUSTRATIONS

The three separate and very different teaching procedures, each referred to by its proponents as 'the discovery method,' can be illustrated by reference to ways of teaching a multiplication rule for quotients. All three of the so-called discovery techniques are in contrast to any approach through initial statement of a rule. Just to get that approach out of our minds let's examine two possible statements of such a rule, one verbal, the other mathematical:

1. For each first number, for each second number not zero, for each third number, for each fourth number not zero, the quotient of the first number by the second number multiplied by the quotient of the third number by the fourth number is the product of the first number by the third number divided by the product of the second number by the fourth number.

2. $\forall a \forall b \neq 0 \; \forall c \forall d \neq 0 \quad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

Such sentences are avoided in traditional textbook approaches by resorting to garbled mixtures of arithmetic and meta-mathematics--rules which abandon references to numbers and tell one how to manipulate figures to get an answer. The very existence of such language as '. . . the product of the numerators over the product of the denominators' traces to confusion between numerals and the entries which numerals represent. Mathematicians have become aware of entities and relations between the entities; they have then named, or symbolized, the entities and described the relations between them; and then they have thought that they were giving us the essence of their discoveries when they gave us the symbols and sentences. In the resulting desperation to make things clearer, they have composed verbal descriptions of their symbols and what they do with them, a sorry substitute for making those with whom they are trying to communicate aware of the concepts and generalizations they themselves have in mind on a non-verbal level. Stating an arithmetic generalization concerning products of quotients requires sentences about numbers. If a mathematician says what he means, he needs sentences of the types illustrated in (1) and (2). In contrast, let us see how discovery lessons based on each of the three interpretations of 'discovery method' outlined above would proceed.

Prerequisite to all three of the approaches are these previous learnings:

1. When a rectangle is divided into six congruent squares, each square is called 'one-sixth of the whole'; or, in general, for each counting number n, if a rectangle is divided into n congruent squares, then each square is called 'one-nth of the whole rectangle'.

2. When a rectangle two units wide and three units long is divided into unit squares, the resulting six squares can be thought of as two of three, or as three of two, that is two rows of three squares each, or three columns of two squares each. That is, for

each counting number m, for each counting number n, a rectangle m units long and n units wide is made up of m of n squares or n of m squares.

3. The number of unit squares in any rectangle of the kind described above is mn, that is, multiplication is the arithmetic process by which one finds the area-measure of a rectangle when he knows its length-measure and width-measure.

Inductive Method

The class reviews the three items above, probably in the reverse order from that in which they are stated above. (It would be a more natural quasi-research approach if a problem to be solved had been stated first, for example:

What is 1/2 of 3/4?

Unfortunately, things are not usually done in this order. Hence, the prerequisite review is imposed authoritatively by the teacher, instead of being something thought of as a way of shedding light on a new, unsolved problem. Conducting a review before stating the problem injects a false and unnatural element into the discovery process, and robs the children of experience in an important technique of research.)

Second, if not earlier, the problem is stated, and the teacher helps the student to find an answer to the question, "What is 1/2 of 3/4?" by making appropriate drawings (Figures 1, 2, 3) and then counting.

First, a rectangle is divided into fourths and three of them are shaded. Next the shaded part is divided into halves, and cross shading is used to identify one-half of three-fourths of the original rectangle. Finally, each part of the cross-shaded section is found by counting to be one-eighth of the original rectangle. Then by counting, the children see that one-half of three-fourths of the rectangle is three-eighth of the rectangle. So, the teacher records on the chalkboard at one side, possibly under the heading "Data":

1/2 of 3/4 = 3/8

Referring back to "two of three" and "three of two" as synonyms for '3X2' and '2X3', all agree that it should be all right to restate the last problem as:

3/4 X 1/2 = 3/8

This process of making and examining appropriate drawings is repeated for at least three more examples, but usually not more than that. By this time the recorded data may look something like this:

1/2 of 3/4 = 3/8 or 3/4 X 1/2 = 3/8

2/3 of 1/5 = 2/15 or 1/5 X 2/3 = 2/15

3/4 of 3/5 = 9/20 or 3/5 X 3/4 = 9/20

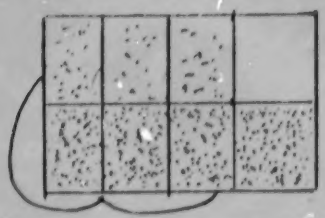
3/7 of 2/3 = 6/21 or 2/3 X 3/7 = 6/21

A teacher alert to the structural properties of the set of numbers of arithmetic will also have, for example:

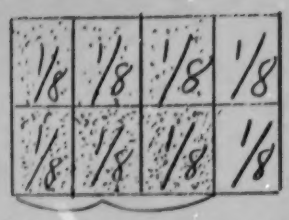
3/4 of 1/2 = 3/8 or 1/2 X 3/4 = 3/8

The differences between the experimental procedure for this exercise and the drawing in 1 to 3 will have been illustrated on the board as follows:

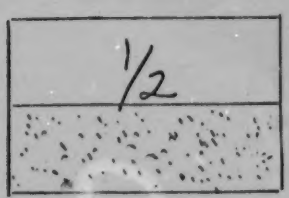
3/4 Figure 1



1/2 of 3/4 Figure 2



1/2 of 3/4 Figure 3



1/2
Figure 4

3/4 of 1/2
Figure 5

3/8
Figure 6

So 3/4 of 1/2 = 3/8, or 1/2 x 3/4 = 3/8, and corresponding entry is included in the table of data.

The next part of the inductive lesson is usually introduced by the teacher with some remark, such as "Now, let's see how we could have found answers just by looking at the fractions." The children are told to examine each pair of fractions and the fraction which names the product, and to try to tell how they could have found the product without drawing the picture. In the inductive method, the ability to "tell how to do it" is the criterion by which the teacher recognizes that discovery has taken place.

As a matter of fact, the children at this stage in arithmetic do not have the linguist machinery for stating precisely what they mean when they have discovered the rule. The inductive method forces them into saying such things as "Well, you multiply the top numbers, and you multiply the bottom numbers, and then you write the top answer over the bottom answer."

A teacher's insistence on more polished statements may call for something like "Well, to get the numerator of the answer, you multiply the number of the first fraction by the numerator of the second fraction; then . . ."

If the teacher is one of those rare ones who is trying to keep clear the distinction between numbers and numerals, he must then go back and remind the student that a fraction is not a number; it is a name for a number. So the student must revise his answer to say that the numerator of the answer is a numeral for the product of the numbers named by the numerators of the two original fractions, etc. All this time, the teacher keeps insisting that "If you can't tell it, you don't know it."

Finally, a statement acceptable to the teacher is achieved. Each student may write the new rule in his notebook, or he may be asked to turn the page in his book, where he will find a rule already stated. (Of course, some members of the class may have looked ahead and found out what they were supposed to be saying.) The students are then considered ready to work a set of practice exercises.

When well done, this method does give each student what is sometimes called 'an intuitive crutch,' a device upon which he can fall back; he can repeat the experiment whenever he finds himself in doubt about the rule. Security and independence are provided, and, in contrast to a tell-them-and-drill approach, the inductive lesson seems a revelation in pedagogy. A well-presented authoritarian approach is very satisfying to someone who is already

aware of the ideas being presented; but for a learner to whom the ideas are new, the authoritarian method at its best is a feeble best in comparison with the inductive method. And the inductive method at its best is a feeble best in comparison with a genuine discovery approach which promotes and recognizes a nonverbal awareness state in creative thought.

The nonverbal awareness method:

Let us contrast the procedure called the inductive method with a discovery lesson which follows more closely a pattern of genuine research. The goal, as in the preceding development, is the "multiplying fractions" rule.

In this approach, statement of a problem must precede the developmental review leading to the physical space drawing experiment by which an answer is found. From then on, the lesson may proceed exactly as in the inductive method up to a certain point. Then a highly significant, sharply defined difference appears. In this kind of a discovery approach, as soon as each child has mastered the drawing-and-counting device for finding products of pairs of quotients, he is given a list of problems to work. As far as the student knows, he is to work every problem on the list by making an appropriate drawing. The class activity thus shifts to individual seat work. What is not done before the bell rings is to be finished for homework. The problems on the list should be arranged to promote interesting observations. The work sheet should provide space for recording the answers to the problems. (At least two pairs of consecutive examples will illustrate the commutative principle for multiplication.) To give pause to the conscientious little task-doer who might work the whole list be making drawings just because that is what the directions (or the teacher) said to do, examples like the one below should be included:

$$\frac{17}{21} \times \frac{11}{19} = ?$$

Students accustomed to this kind of discovery approach will, of course, be on the lookout for a short cut. Furthermore, from previous experience with the course and their teacher, pupils will know that finding and using a short cut is something that will be highly approved. Until this experience has been built up, the directions for the exercise must contain some challenge, such as "If at any time you think you know what an answer is going to be before you make the drawing, please write down the answer and raise your hand." When a hand goes up, the teacher should go to the student, congratulate him, or advise him to check his hunch by another experimental drawing. It must be pointed out, however, that this direction injects a false note into the stage setting for true research. It is simply a device by which a teacher helps a class to get its first experiences with this kind of learning.

In such a lesson, how does the teacher know when a discovery has taken place? Suppose he is walking about the class looking over shoulders. A sudden start, a flush of excitement, and a student begins writing answers almost as fast as he can put them down. When a student begins to write correct answers without making the drawing, we have evidence of the advent of awareness. Awareness of what? Awareness of the "multiplying fractions" rule.

If the child were not aware of it, he couldn't be using it, to call a halt in his progress at this point for a long, tedious verbalization attempt is damaging in several ways. In the first place, it belittles the exhilarating accomplishment of the actual discovery. Whether he can "tell it" if not, the child knows very well that he has something.

Then, in the second place, the verbalization attempt at this grade-school level is doomed to frustration. Numerical variables, universal quantifiers, and the zero restrictions on domains are all necessary to a precise formulation of the generalization the child has just discovered. The most that a teacher can do is to accept a garbled verbalization of what was at the beginning a clear, dynamic insight. The more sensitive a child is to precision in the use of his mother tongue, the more damaging this process can be to him and to the learning which a moment before was so clear on the nonverbal awareness level. The more skillful the child is in interpreting his mother tongue, the higher the probability that a literal interpretation of his own incorrect statement will impinge itself upon him. If this

literal interpretation of his incorrect statement contradicts the awareness that was already there, the original learning is mutilated. In a lesson of this kind, the child who notices that his neighbor is writing correct answers without making drawings is electrified into a search for "what's going on." Knowing that one of his classmates has discovered a short cut has a much more highly motivating effect upon him than a search for short cuts suggested by the teacher. The child who does not discover the rule before the end of class period still has his chance during his homework period--if well-meaning parents do not step in and spoil it all for him!

On the following day the teacher presents other examples, such as:

$$\frac{5}{3} \times \frac{7}{8}$$

$$\frac{6}{5} \times \frac{9}{4}$$

$$\frac{6}{2} \times \frac{21}{7}$$

In other words, over generalization and incomplete generalization are prevented by the introduction of examples which correct possible false impressions, not by verbalizing the rule at this stage of the game and conducting a sophisticated search for false instances. (The more sophisticated procedure is available and appropriate a few years later when these same children will have acquired the necessary linguistic equipment.)

I do not wish to belittle the tremendous improvement in instruction which results from pursuing the inductive method, in contrast to authoritarian procedures. But those who think teaching for discovery is the inductive method are still blinded to this second tremendous breakthrough in theory of instruction. It is recognition of the nonverbal awareness stage in inductive learning that converts the classroom experience into that of actual discovery, the kind of thing that promotes a taste for and a delight in research.

The incidental method:

In contrast to both of the instructional procedures outlined above, let us think about the teacher who trusts that the"multiplying fractions" rule will be discovered incidentally.

In one project a child will need to find the cost of a sheet of gold leaf to line a jewelry box, for instance, when he knows the cost per square foot and knows that the sheet must be one-half foot wide and three-fourths foot long. He is taken through the drawing experiment until he finds an answer. No other instances of multiplying one quotient by another come along in that project. Six weeks later (or six months, or a year) in connection with another project, he has another occasion to find the product of two quotients. Again the physical space drawings are used, the answer is found, and the project proceeds. All pressure from teacher and group is directed toward finishing the project. (Anyone who becomes intrigued by the multiplication process itself and digresses to investigate will be frowned upon by many of the educators advocating this kind of schoolwork). From such examples widely distributed over time and space, the probability that the generalization will emerge is almost zero. Only a little genius who has somehow escaped with his taste for contemplation undestroyed will recall such widely scattered experiences all at the same time, and hence become aware of the mathematical generalization underlying them. Critics of education who confuse this kind of school program with everything else referred to as 'learning by discovery' are justified in opposing a discovery method. Much as I, the writer, abhor authoritarian presentations of sentences which express generalizations, the words later to be clarified by examples, I would prefer falling back upon that relatively leaily and deadening procedure, rather than to rely upon important basic principles "coming clear" from experiences incidental to so-called 'project activity'.

So much for the status of the discovery method in educational parlance and educational literature at the beginning of the decade to be known as ' the sixties'. It is high time

however, for another category to be added to the list when discovery in mathematics is under discussion. Even the instructional procedure described under the nonverbal awareness method is not properly called 'the discovery method'. It is possible to arrive at discovery by pure deducation, that is, by manipulating sentences according to logical rules. It is not likely, but neither is it impossible, that the "multiplying fractions" rule would be discovered by derivation; most deductive proofs are checks for sentences first arrived at inductively; but some very important principals both in mathematics and in physical science, have been discovered by deduction. A good example of something in secondary-school mathematics usually discovered (that is, re-discovered) in this way is the rule for finding the measure of one side of a triangle in terms of measures of the other two sides and the cosine of the measure of the included angle. This is a case in which one begins with some sentences which express observations of a drawing. Starting with these sentences as premises, pure deduction takes over and out comes the desired formula. This last sentence, discovered deductive is then interpreted in terms of how the varibles were used in the original drawing. This interpretation of a final sentence in the deductive sequence reveals the desired geometric rule. This is a discovery process just as truly as is the inductive leap that takes place in the nonverbal awareness type of discovery. How do we know both are discovery? Because a person can do them all by himself. The thing learned is not received by communication.

Much of the confusion permeating discussions of learning by discovery comes from failure to distinguish discovery from communication. One does not discover a concept or a principle in a book. He may discover the name of a concept, a definition, a sentence which states a principle, etc., but before he "has" anything, he must engage himself in the process of interpreting the words and sentences of the author, that is, he must complete a communication process. Communication usually plays an important part in setting the stage for discovery in the teaching of mathematics, but that part is quite different from actually communicating the new thing to be learned.

LEARNING BY DISCOVERY: WHAT IS LEARNED? *

Bert Y. Kersh

It goes almost without saying that discovery learning is in vogue today, especially where mathematics and science educators are concerned. This observation is supported by the substantial curriculum efforts in mathematics and science (1, 2, 3, 4)+, and by the programmatic research endeavors on cognition (5), not to mention the numerous independent efforts by researchers over the country. The reason for this is not entirely clear to me, however, so I would like to consider what is known about discovery learning that has some basis in experimental research. Specifically, I will attempt to answer the question, "What does the student learn when he discovers a principle or generalization in mathematics independently that he may not learn when taught these same principles or generalizations directly?" Then I would like to suggest some practical implications.

It has been impressed upon me since I first began studying discovery nearly 12 years ago that, as a learning process, it means all things to all people. Max Beberman, for instance, defends the use of the discovery method in the UICSM curriculum on the grounds that ninth graders are attracted to the "what-would-happen-if" question, regardless of its practicality from the adult point of view. The fact that the learner can discover for himself is all that he needs to justify mathematics. In Beberman's words

"...the discovery method develops interest in mathematics, and power in mathematical thinking. Because of the student's independence of rote rules and rountines, it also develops a versatility in applying mathematics"

At least with respect to Beberman's claim that the learner is naturally curious and intrigued by a challenge to find out "what-would-happen-if," his justification for the discovery method has some basis in research. In other words, there is research evidence to indicate that students are motivated to continue learning in a subject matter area when forced to discover generalizations for themselves (6, 7). However, very little, if any, empirical evidence can be offered in support of their other claims.

Actually, researchers do not typically concern themselves with such phenomena as "power in mathematical thinking" and versatility in application of a subject matter area. Seldom do we find a research report of a study of discovery processes which measures much more than the learner's ability to retain what was learned.

Presumably, what Beberman and others who teach mathematics are concerned with, more than whether the student remembers a particular formula or proof, is whether the learner learns some strategies of thinking which will stand him in good stead when next he is faced with a learning task in mathematics related to the one dealt with in the classroom. This is not to say that teachers of mathematics are unconcerned with the student's memory for general laws and principles. Rather, it would be more correct to say that the concern is that the student understand whatever he learns in hopes that with understanding will come the ability to reconstruct a generalization from memory even after it is forgotten, and the ability to apply a general law in new tasks wherever the generalization may be applicable.

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* The Arithmetic Teacher, April, 1964, pp. 226

+ Numerals in brackets refer to the references at the end of the article.

Learning By Discovery: What is Learned?

To summarize the claims for learning by discovery, then, we may say this: the claim is that when the student learns by independent discovery he (a) develops an interest in the task, (b) understands what he learns and so is better able to remember and to transfer what is learned, and (c) learns something the psychologist calls a "learning set" or a strategy for discovering new generalizations.

As earlier stated, there is research evidence to support the claim that learners become more interested in whatever they are learning, but the evidence is equivocal with respect to the claim for increased understanding, and practically nonexistent with respect to the third claim concerning the attainment of complex learning processes. The only study that has come to my attention concerning the latter deals with problem-solving patterns employed by high school and university students working with problems of elementary arithmetic (8). Buswell found that high school and university students are remarkably inconsistent in employing those problem-solving patterns which teachers of elementary arithmetic sometimes try to teach directly. I am referring to such simple procedures as estimating and checking answers.

Before proceeding further, it may be well to define the term "discovery." As is the case with defining programmed instruction, it is difficult to find a clear definition of discovery. In the research literature, the term discovery frequently describes a learner's goal-directed behavior when he is forced to complete a learning task without help from the teacher. This is the way I am using the term.

If the learner completes the task with little or no help, he is said to have learned by discovery. The most significant teaching variable is the amount of guidance or direction provided by the teacher during the discovery process. In practice, considerable help from the teacher may be provided and still the learner may be said to have learned by discovery, but in such instances the process is usually qualified and called guided discovery. As the amount of help from the teacher increases, it is said that opportunities for discovery decrease and the learner may rely more on rote processes of learning.

Actually, the terms "discovery" and "rote" are probably not "opposite" terms at all. They may, in fact, be interdependent in the learning process. It may be more correct to confine the one term, discovery, to that phase in learning which precedes the learner making the response desired by the teacher for the first time (the problem-solving stage) and the other, rote learning, to the phase which follows when a learner is memorizing or acquiring increasing skill.

It should be noted that the above definition is stated in operational terms; i.e., it states how the teacher operates with the learner or the conditions under which the learner operates during discovery. The definition does not infer anything about the learner's sensations or perceptions during or at the point of discovery. When the learner first makes the required response with little or no help from the teacher he may or may not have experienced "insight" and he may or may not understand. Also, he may have been acquiring some other skills quite incidentally in the discovery process. Similarly, even later when the learner is memorizing the required response or practicing it, he may still ascertain something new (to him) about that which he is practicing.

Returning now to a consideration of what is learned when the learner employs the discovery process, it has been noted that the research evidence is quite inconsistent in this respect. However, it is becoming increasingly more evident that the somewhat inconsistent findings by researchers in recent years may actually reflect different learning outcomes resulting from two or three quite different processes of learning by discovery. We may no longer speak of the discovery method in general terms and make all-encompassing claims about it any more than we can speak of the method of teaching by machines as if it were one technique.

Learning by Discovery: What is Learned?

To better communicate this, let me direct your attention to the tasks employed in at least six of the recent research studies. By the tasks, I mean that which a learner is asked to discover and under what circumstances. From the tasks, we may infer what the learner might have learned in addition to what the researcher measured, and it will be shown that the apparently different findings of some of these studies are actually highly consistent and that it is possible to make some new interpretations of the research findings.

One task, which I will call the "Word Task," is to select the one word from among five that does not belong with the others of the set. The remaining four are orgranized according to some principle. For example, a coin, a plate, a button, a ball and a wheel are all round, in one sense, but all except the ball are also flat. Consequently, ball is the odd word. The principle involved is that four words name objects that are similar in some way and the fifth word names an object that is different. As many as 60 sets of words have been used as learning material, involving perhaps 20 different principles.

Since 1955, the Word Task has been used by Craig (9) and by Kittell (10). The learning took place in a group setting. Each subject was instructed to underline the word that did not belong with the others of the set and was informed whether or not he was correct immediately upon selecting a word. If right, he went on to the next problem set; if wrong, he tried again. He was instructed to guess if he did not know the answer. Of course, the amount of guidance given to the learners in their efforts to discover the odd word in each set was varied.

In effect, then, the subjects were set by the instructions and by the feedback received during the learning process to locate the odd word, not necessarily to search for the underlying principle. Furthermore, they had little opportunity to indulge themselves in the process of discovery because each of the five-word items was accomplished quickly or passed over unresolved. The more difficult items could be answered by process of elimination, if necessary.

By way of contrast, let me describe another task which I personally have used on three occassions (6, 7, 11) I shall call it the "Arithmetic Task." The subjects are asked to learn one or two rules of addition, each of which is relatively novel so far as the subjects are concerned. One of the rules to be discovered is that the sum of the first N odd numbers is equal to N-squared.‡ The other rule is actually the more generally applicable algebraic formula for summing any series of numbers having a constant difference between numbers. In my studies the subjects were dealt with individually and in the study by Craig a group-pacing technique was used.

Again, varying amounts of guidance are provided. At one extreme, the "discovery group" is instructed to find a "quicker" or "different" way of summing the series. At the other extreme, the "rule-given" or "rote learning group" is told the rule and instructed to practice it on the examples provided. Consider now the discovery group. In my experiments, at least, the experimenter asks the learner to "think aloud" or to write down his responses at every step of the way. The experimenter is present at all times, giving encouragement continually. His main purpose is to keep the learner going in hopes that he will eventually arrive at the correct solution.

In short, the two tasks in question differ essentially in that the Word Task orients the learner to the answer and the Arithmetic Task orients the learner to the process of discovering the answer. The Word Task employs the discovery method rather inefficiently as a means to the end; the Arithmetic Task treats the discovery method as an end in itself, with its own rewards and incentives. The learner is taught to search for answers, not to provide them.

‡For example, the sum of the first three odd numbers is three-squared or nine.

This comparison is, I believe, significant in view of the somewhat contrasting results of the studies involving the two different tasks. The studies employing the Word Task generally report results indicating that subjects learn more initially when they do not have to discover for themselves. With respect to retention and transfer, the evidence is not clear-cut.

The studies employing the Arithmetic Task also favor the more directive procedures during initial learning. In fact, many subjects may fail to discover the more complex rule at all within the time limits of the learning period. However, when these same students are retested a month or six weeks later, it appears that they have learned something after all. Individuals in the groups taught the rules directly tend to be quite forgetful by comparison.

Judging from the tasks involved, the seemingly equivocal results reported by Craig, Kittell, and Kersh may actually be reflecting different learning outcomes resulting from two quite different processes of learning by discovery. In the case of the Arithmetic Tasks, the subjects appeared to develop an interest in the task which carried them on to independent study. With the Word Task, the subjects learned to achieve a specific objective and then stop.

Haslerud and Meyers (12) have also reported research bearing directly on the topic area. However, they used a different task, a coding task similar to the cryptogram "Come to London" in the Stanford-Binet. Each learner was required to derive the codes for ten such problems solely from concrete instances, and ten more specific directions for deciphering the code. As would be expected, the learners were able to decipher more of the codes when the rules were given than when the codes had to be "discovered." However, one week later, on a transfer test, the learners showed greater gains on the rules which had been discovered than they did on those which had been told to them. The authors state that their results give strong support to the postulate of Hendrix that independently derived principles are more transferable than those given. It may also be that their subjects became more interested in the codes which they discovered and practiced them a little more.

I am trying to direct your thinking in ways that to some of you may seem devious and involved. Others of you may find this line of thinking as fascinating and rewarding as I do.

Let me ask the question we started with once again and attempt to answer it in the light of what has been said about the research evidence. The question is, "What do you learn by the discovery process?"

The answer is, you learn essentially only what you practice during the process. You may "learn how to learn" other mathematical generalizations only if you are exercised in techniques for developing generalizations. You may learn to apply a specific principle effectively only if you practice making applications. The benefit of learning by discovery comes from the fact that sometimes (but not always) the learner may engage in greater amounts of practice in employing problem-solving strategies and in making applications than he would by some other teaching-learning process.

By this token, you would have to conclude that learning by discovery is not necessarily the most effective learning procedure for all teaching objectives and that what benefits there may be can be explained rather simply in terms of practice and reinforcement. This is exactly what I would have you conclude--with the one important exception. Be reminded that the learner does frequently develop a greater interest in what he is learning when forced to learn without help. This rather mystical motivating power is unique to discovery learning, and it is most fascinating to consider.

In writing of this motivating power, a number of researchers (4, 13) have referred directly to the motivational concept of "competence" which has been developed by White (14). White postulates that exploration, manipulation and mastery are intrinsically rewarding. Consequently, human beings are motivated towards competence. An important aspect of White's notion is that the rewarding features of this elusive drive lie in the arousal and maintaining of activity, not in the completion of the act. At the same time, the learner must have success experiences with any specific activity. Otherwise, he may come to avoid that particular activity for loss of interest or fear of failure. This "power in mathematical thinking" which Beberman speaks of, which may be the same as the "intellectual potency" that Bruner describes, may be more a function of the latter than the former.

What do these findings mean to the teacher? It is evident that one can learn different things by the process labeled "discovery." Therefore, to talk intelligently about discovery methods of teaching or learning, one must state his desired outcome or teaching objective.

For the goal of teaching specific aspects of subject matter--aspects which could be either memorized or discovered--it is clear that the process which was described as "directed learning" is the most efficient. Generally speaking, a teacher should not employ highly directed techniques if she wishes to develop long-term retention and transfer effects. However, the directed procedures do not necessarily produce learning outcomes which are short-lived. Retention and transfer effects primarily reflect practice and reinforcement schedules.

With organized bodies of information, very often the teacher is most interested in teaching the organizational framework itself. Usually this framework is in the form of principles, rules, generalizations, and conceptual schemes. If so, the research evidence suggests the intermediate direction or guided discovery techniques. It makes little difference, apparently, whether the rule or principle is discovered or is taught directly, provided the learner is reinforced for effective practice in using the rule or principle. Organizational schemes, however, often need to be understood through the establishment of relationships between new and previous learning. When the learner lacks the necessary background of information and is not motivated, generally it is more effective to establish such relationships by using techniques of guided discovery rather than more or less directed techniques. The important thing is that the new learnings are established in relationship to previous learning. Provided the learner has the related knowledge and willingness to assimilate the material, the directed technique may be equally effective and certainly less time-consuming than guided discovery.

Very often the teacher has as his objective techniques of discovery, per se. The actual subject matter involved may be of secondary importance. The purpose of the learning experience is to exercise and to reinforce the learner in what may be called "searching behavior"--strategies of problem solving, divergent as opposed to convergent thinking, flexibility in thinking--in essence, the characteristics of what is often labeled "the creative person." With such objectives, discovery or guided discovery techniques are most appropriate. However, if the task is so difficult that the learner does not succeed in discovering the relationships he is supposed to discover, there will be very little opportunity for reinforcement of that very process which is being taught. It is most important that the learner have success experiences when learning by discovery.

It has become increasingly apparent that the learner's attitudes toward a subject-matter area may be as important as what he learns in the cognitive sense. If a student is highly interested in a subject, he is likely to continue to learn. Under appropriate

conditions of practice and reinforcement, the discovery technique will foster favorable attitudes and interests. It is interesting and challenging for students to discover, particularly if their efforts are successful, or at least occasionally so. The opposite effect may result when their efforts never or almost never meet with success. Also, there is reason to believe that an intermittent or regular pattern of success may be more effective than a regular pattern in maintaining interest.

SUMMARY:

Consistent throughout the recent research, as interpreted above, is evidence that the discovery method is effective for what it requires the learner to do and for what is reinforced during learning. The learner may acquire more effective ways of problem solving through the discovery process than through another process simply because he has an opportunity to practice different techniques and because his more effective techniques are reinforced. Similarly, the learner may become more proficient in applying rules through the directed process of teaching simply because, through formal practice, he has more opportunity for effective practice and reinforcement than otherwise. Guided discovery seems to offer a happy medium between independent discovery and highly directed learning. Some of the efficiency of directed learning is maintained along with the benefits of the discovery process, specifically, motivation and problem-solving skill.

Learning is a complex process. It cannot be explained solely in terms of practice and reinforcement. However, these two concepts are powerful and do enable us to understand much of what a student learns by discovery.

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Learning by Discovery: What is Learned?

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LEARNING BY DISCOVERY: INSTRUCTIONAL STRATEGIES *

What characterizes the discovery approach? What difference does it make whether the learner "invents" a principle of mathematics or simply is told the principle? The expected answer would be that learning is more "effective," even though it may take the learner longer to learn by discovery. What advocates of discovery learning generally mean by effective learning is not often spelled out in detail. However, many would agree that when the student learns by discovery he (1) understands what he learns, and so is better able to remember and to transfer it; (2) he learns something the psychologist calls a "learning set," or a strategy for discovering new principles, and (3) he develops an interest in what he learned.

It is becoming increasingly more evident that the somewhat inconsistent findings by researchers in recent years may actually reflect different learning outcomes, resulting from two or three quite different processes of learning by discovery. Moreover, if people in a group were each asked to give a concrete example of the discovery approach, I would expect a number of different responses. In the final analysis, it is highly probable that we do not all have precisely the same idea of the discovery approach after all.

Research has indicated that there are subtle, but highly important, differences in the way teachers guide the learning process when students are attempting to "discover" mathematical principles.

Let's start with an example of teaching a mathematical "principle" by the discovery approach. The principle may be stated as follows: The sum of the first n odd numbers is n^2. For example, the sum of the first four odd numbers (1,3,5, and 7) is 4^2 or 16. This particular principle has been used frequently in educational research because, like many mathematical principles, it can be learned in a variety of ways. For example, the rule may be taught as if it were an isolated fact, or it may be interrelated with other information which the learner already knows. For instance, a student may learn to explain the principle in terms of the symmetry of the series, or he may learn to explain the principle geometrically in terms of the area of a square, the side of which is n units in length.

Imagine a learner at his desk working an exercise consisting of ten addition problems. Each problem is a series of odd numbers beginning with one (for example, 1+3+5; 1+3+5+7; etc.). He has been instructed to find a shortcut for summing each series.

Now, ask yourself, "What are we trying to teach in this exercise?" As advocates of discovery learning, we should say we want to teach (1) the mathematical principle, (2) a procedure for identifying mathematical principles from examples, and (3) a desire to learn more about numbers in series. Why are we using the discovery method? Because we feel the learner will attain all three objectives through discovery.

Instead of saying that the mathematical principle is taught, a researcher might say "answer-giving" behavior is taught to the learner. Also, instead of saying that a procedure for identifying mathematical principles from examples is taught, he might say that we teach "answer-seeking" behavior. Answer-giving behavior characteristically involves information processing which produces a solution to the problem at hand, in this case, the shortcut n^2. When the learner performs arithmetical operations, replaces numerals, and factors, he is engaging in answer-giving behavior. The end

*From: The Arithmetic Teacher, October, 1965, pp. 414-417.

Learning By Discovery: Instructional Strategies

result, or product, of his behavior is an answer, right or wrong. Answer-seeking behavior is different in that it deals exclusively with actions the learner takes in preparing for answer-giving. The result is a plan of action or a decision about what to do next. It involves separating out the various features of a mathematical expression and deciding which to "focus upon" in answer-seeking behavior; deciding what kind of information will be needed to accomplish the task at hand, and what to do with the information once it is obtained is answer-seeking behavior. In short, answer-seeking behavior is deciding what to do, and answer-giving behavior is actually doing whatever has been decided.

Let us return to our hypothetical student. You are sitting with the learner, watching him in his efforts to find the short-cut. He is having difficulty, but you are resisting the temptation to help him, because the discovery method dictates that you withhold instructions. However, does this mean that you must avoid interacting with the learner at all? Certainly not. A teacher's job is to facilitate learning.

A careful analysis of the learning task in this case would indicate that we can interact with the learner in at least two different ways. Each way is characterized by withholding instructions and, at the same time, assisting the learner in his efforts to achieve an instructional objective. This is possible because there is more than one instructional objective. Let us consider each of these two discovery methods separately.

PROVIDING "ANSWER-GIVING" INSTRUCTIONS AND WITHHOLDING "ANSWER-SEEKING" INSTRUCTIONS:

One discovery approach results when the teacher guides the learner by revealing the mathematical principle through "hints" which are provided one at a time. Skillfully employed, the procedure effectively leads the learner into pathways of meaning and understanding where he otherwise might not venture or which he might simply overlook. The worst that can be said for it is that it is comparable to dangling a carrot before a donkey in an effort to make him move. At best, the learner will complete the task with a clear and meaningful knowledge of the principle involved. Unless he is skillfully taught, however, the learner may as well have been told the answer in the first place with little or no explanation of it.

For example, the learner might be told to look at the symmetry of the series and to notice that the numbers following the midpoint increase, just as those preceding the midpoint decrease in magnitude. If this does not prove successful, the instructor might suggest that the learner find the arithmetic mean of the series and compare it with the "middle numeral." The procedure almost guarantees that the learner will find the answer during the course of instruction. On the other hand, he may or may not improve his ability to seek answers. He may even learn undesirable approaches.

GIVING "ANSWER-SEEKING" INSTRUCTIONS AND WITHHOLDING "ANSWER-GIVING" INSTRUCTIONS:

This procedure is the opposite of the first. Although the teacher may not give the learner any hints which reveal the answer to him, the teacher is permitted to suggest alternative plans of approach in problem-solving strategies. In other words, the teacher may facilitate the discovery or problem-solving process without revealing the principle itself. Mathematicians employ a variety of problem-solving strategies, some of which are quite formalized and others of which are best described as informal approaches. "Persistence" and "flexibility in thinking" are examples of the latter.

Learning By Discovery: Instructional Strategies

Imagine that our learner is a junior high school student who is just beginning algebra, and we are interested only in fostering the informal problem-solving techniques. The instructional strategy is to ask the learner to report what he is thinking in the process of working the problems. We ask him to "think out loud." Whenever we determine that the learner is being persistent and flexible, we will praise his efforts and offer encouragement. Whenever the learner appears to persevere in one problem-solving approach, we instruct him in how to proceed on a different tack, hopefully one which will lead more directly to the solution. Finally, if the learner shows signs of giving up, we will persuade him to continue in hopes that his efforts will soon be rewarded. All this help and encouragement is to be provided regardless of how successful the learner is in finding the shortcut.

A specific set of instructions designed to suggest a different strategy to the learner, without indicating the answer, might be, "try comparing the problems on your answer sheet. See if you can benefit by what they have in common and what is different about each of them." This instruction tells the learner how to begin processing the information before him, and hopefully produces some observations about the number series which will suggest the shortcut. The approach is generally a good one. Your next instruction may be, "Decide what is the most important element that these problems have in common. Think about this one feature, and tell me how you think it might help you find the sum."

Instructions such as these do not reveal anything to the learner about the shortcut he is seeking. Rather, they channel his thinking in a way which will increase the probability of his finding the answer. There is no guarantee that he will, however. From what we know about human behavior under these circumstances, it is likely that the learner will not find the shortcut while working in your presence. Since we have told him essentially how to approach the problem and let him know when he has employed good answer-seeking behavior, it would seem more than likely that the learner will profit from this aspect of his experience. In other words, whereas the first instructional strategy almost inevitably will teach the mathematical principle involved, the second one almost certainly will teach a technique for identifying principles--one which applies to other problem-solving situations as well.

COMPARISON OF THE TWO APPROACHES:

Each of the two approaches described are properly identified as discovery approaches, because the instructor withholds information and the learner must find it out for himself. These two approaches differ only in the type of information provided and withheld. Both approaches could teach the mathematical principle and the skill in identifying principles. However, each method emphasizes a different instructional objective, specifically that which relates to the information which the teacher provides the learner. We cannot be at all certain that the learner profits much from the "unguided" experience he has. We can conclude only that some learners profit from unguided experiences and others do not.

We have considered two instructional strategies and discussed them with reference to the first two of three objectives identified earlier. The objective which has not been discussed is that which pertains to developing interest in mathematics. Max Beberman defends the use of the discovery method in the UICSM curriculum on the grounds that learners are attracted to the "what-would-happen-if" question, regardless of its practicality from the adult point of view. The fact that the learner can discover for himself is all the learner needs to justify mathematics. Actually, there is very little evidence in the research literature that bears on this reaction to discovery learning.

Learning By Discovery: Instructional Strategies

Researchers have attempted to measure the emotional reaction of the learners to discovery-type tasks in various ways. The results of some indicate heightened interest. Others did not indicate increased interest.

Personally, I believe that this phenomenon of interest which sometimes results from discovery learning is a cognitive motive which might be related to a drive for mastery or simply reflect our native curiosity. The drive to master a task sometimes is aroused when we are engaged in a task requiring the manipulation of objects or ideas. This drive is particularly evident in the behavior of young children who may persist at a manipulative task for long periods of time. Curiosity, on the other hand, may be aroused in situations which have elements of surprise, perplexity, or doubt. In our number series example, for instance, many learners appear to hit upon the solution unexpectedly. This surprise reaction may tend to pique their curiosity. Undoubtedly, many of you have used effectively the technique of "torpedoing" an hypothesis which your students offer as conclusive. Demonstrating the vulnerability of some mathematical principle produces doubt in the student's mind, which in turn results in some further exploration and study. Perplexity or "puzzlement" may be produced by reviewing the rich associative structure of an apparently simple principle. Each of these techniques is often involved in the discovery approach.

We may conclude that interest could be aroused by either of the strategies outlined previously. However, the extent to which interest is aroused may be a function of skillful questioning and ordering of experiences for the learner. It may also depend on whether or not the learner is successful in discovering the shortcut as well as upon some rather complex characteristics of the learner about which we know very little. In any event there is certainly no guarantee that the discovery approach will result in increased interest in the task at hand.

CONCLUDING STATEMENT:

Two instructional strategies, both of which involve discovery experiences, but which may be expected to produce different learning outcomes, have been outlined. One fosters answer-giving behavior, the other answer-seeking behavior.

What the student learns probably is related more to the instructions which the teacher provides than to those which he withholds. If instructions on how to seek alternative ways for producing answers are provided and instructions on how to carry out the plans are withheld, we may expect that the student will learn how to seek answers in general, but may not actually learn the answer he is seeking. An instructional strategy must produce the behavior which is to be learned; that is, the learner must do what you want him to learn as best he can so you can act upon the imperfect performance in an effort to help the learner perfect it. By selectively withholding instructions, the teacher may cause the learner to engage in the kind of behavior the teacher is interested in, but instruction may not be effective until the teacher provides instructions designed to guide behavior in the desired direction.

The two different instructional strategies described elicit different behaviors for the teacher to act upon. Consequently, the learning outcomes may be different. Perhaps it is time for us to analyze rigorously our instructional techniques to identify the particular behaviors each technique may be expected to elicit. In this way, we may be able to predict the learning outcomes of our instruction more accurately. Also, it may result in our becoming more critical of those who label instructional strategies "the discovery method" or "the lecture method" and who evaluate them as good or bad accordingly.

FLOW CHARTING THE COMPONENTS OF TEACHING STRATEGY

William R. Arnold

In attempting to improve the teaching of mathematics, educators have developed numerous approaches. Consequently, a large number of materials, textbooks, and teaching procedures have been made available. The new developments are exciting; many of them are helpful. To some extent, as new approaches have been developed and research has been done to determine their relative effectiveness, little information has been made available to teachers with regard to how they can or should alter their specific classroom behavior as the need arises.

Regardless of the approach used, a teacher makes a sequence of moves which determine what pupils do during a mathematics lesson. The nature of instruction provided and the tasks assigned basically determined what the teacher wants the pupils to learn. The environment in which learning occurs can facilitate or hinder learning. Collectively, learning; teacher moves; environment; and examples, exercises, or problems are the basic aspects of a teaching strategy about which a teacher must make decisions before a lesson is presented. When the instructor considers, quite specifically, a series of questions—what the pupils should learn; where the lesson will occur; what materials will be used; what moves will be executed; what examples, patterns, or exercises, if any, will be used——he is able to develop a unique teaching strategy for each lesson.

The approach reflected by a strategy will vary as the teaching situation changes. In all cases a strategy that is developed in this manner emphasizes a teacher's unique competencies. The purpose of this article is to discuss a functional model for developing teaching strategies. The lesson plans which reflect teaching strategies are written in flow-chart form, simple enough to be developed in the amount of time normally available for planning and still provide adequate cues to instruction.

When a teacher enters a classroom there are six types of moves he can utilize. Five of these should directly enable children to learn. The sixth move deals with classroom management. While this is not noted on the flow chart, it is assumed the teacher will manage children in a humane and sensible manner, one that will not make them feel less adequate as human beings. To facilitate learning the teacher can make an exposition move, an illustration move, a demonstration move, a discussion move, or an exploration move. By varying the utilization of moves and the sequencing of moves the teacher can systematically and creatively attempt to facilitate learning. A brief description of the moves is provided below. There is no implicit significance in the order in which they are discussed.

EXPOSITION MOVES

The teacher is making an exposition move when he is explaining an idea or a procedure. A lecture is the classic example of exposition. That is, the teacher giving information verbally. The assumption in using an exposition move is that the child can learn by listening and subsequently reinforcing what he hears. This type of move

*From Schminke & Arnold; Mathematics is a Verb, Dryden Press, 1971

Flow Charting The Components
Of Teaching Strategy

William R. Arnold

has been soundly criticized in educational literature. Yet the notion that people do not learn by listening is ridiculous. What needs to be understood is that exposition moves should not be used all the time or over long periods of time with young children.

ILLUSTRATION MOVES

The teacher is making an illustration move when he is showing the children a physical model, a drawing, a picture, a filmstrip, or motion picture. There may be some nominal accompanying talk but the emphasis is on the visual presentation of an idea or procedure. The assumption in using an illustration move is that the child can learn by looking and subsequently reinforcing what he has seen.

DEMONSTRATION MOVES

The teacher or a pupil is making a demonstration move when he is actively explaining and showing a procedure in a related manner. The assumption in using a demonstration move is that the child can learn by concomitantly listening and looking, and subsequently reinforcing by imitation the technique that has been demonstrated.

DISCUSSION MOVES

The teacher is making a discussion move when he is asking questions and encouraging students to ask questions. That is, there is verbal interaction of an inquiry-response type between the teacher and the children. This type of move is known to be effective for motivating topics, review, discovery, and clarification of ideas.

EXPLORATION MOVES

The teacher is making an exploration move when he gives directions which outline a task children are to undertake. Exploration moves either terminate direct teacher control of a learning situation and set children to work independently, or cause children sequentially to complete tasks in accordance with the teacher's instructions. The tasks may be as simple as reinforcing an idea by completing a worksheet, or as complex as solving a difficult problem or conducting some research. This type of task is designed to extend the pupils' ability to function. In another sense an exploration move may be designed to collect samples of a child's work in order to diagnose where to begin or how to continue a given topic, or to determine the extent to which certain objectives have been achieved. Exploration moves are very complex; it is absolutely necessary to use them because they determine and shape what a child is actually permitted to learn. Since what a child actually learns is determined by himself, the extent to which exploration moves are imaginative, creative, and suited to the intended learning shapes the extent of what the pupil learns and the extent to which this coincides with what the teacher had in mind.

Flow Charting The Components Of Teaching Strategy

William R. Arnold

CLASS MANAGEMENT MOVES

The teacher is making a class-management move when he is using commands or gestures to manage pupil behavior. These involve noninstructional actions such as going to recess, taking lunch count, determining time and place of instruction, and disciplining a child. The nature of class-management moves that are utilized determines the extent to which a child is free to do things.

In developing a flow-chart lesson plan learning is stated in terms which suggest behaviors the child should subsequently demonstrate to indicate that the desired learning occurred. This is important because it suggests sequences of moves to the teacher which are likely to succeed and helps determine the examples or patterns to be utilized. Further, it increases the probability that what the teacher wants the pupils to learn and what they actually learn will coincide. For example, a teacher might want to present a lesson on the distributive property. The phrase, "using distributive property to do written computations involving one and two digit factors," suggests what the lesson should accomplish and the level for which it is appropriate. An another example the statement, "the child should understand indirect measurement," does not suggest a specific focus for a lesson. The statement, "the child is able to use proportions to measure heights indirectly," provides a better focus for a lesson. An objective for a lesson, then, is a specific statement of what the child should be able to do.

During the planning of a lesson care should be given to the selection of examples, exercises, or problems which will be used to promote learning. To illustrate, suppose a group of pupils is to learn how to do written multiplications involving two-digit factors. The examples used for demonstration purposes and the exercises subsequently used for reinforcement should accommodate the range in ability of the pupils and still facilitate learning. Some examples which could be utilized in this lesson are:

(1.) $N = 13 \times 21 = (10 + 3)(20 + 1) = (10 \cdot 20 + 10 \cdot 1) + (3 \cdot 20 + 3 \cdot 1) = 200 + 10 + 60 + 3 = 200 + 70 + 3 = 273$

(2.)
$$\begin{array}{rll} 56 & \sim (50 + 6) & \sim 3000 \\ \times\ 64 & (60 + 4) & 360 \\ \hline & & 200 \\ & & 24 \\ & & \hline 3584 \end{array}$$

(3.)
$$\begin{array}{r} 16 \\ \times 12 \\ \hline 12 \\ 20 \\ 60 \\ 100 \\ \hline 192 \end{array}$$

(4.)
$$\begin{array}{r} 97 \\ \times\ 85 \\ \hline 485 \\ 776 \\ \hline 8245 \end{array}$$

There are noticeable differences between these examples. Each one illustrates a procedure for multiplying two numbers. Consequently, the particular examples used by the teacher would depend on how the lesson was motivated. Practice exercises should be provided that are at least as easy as the third example and as difficult as the fourth example. One way to provide for individual differences is to let each pupil in the group complete some, but not all, of the assigned exercises.

Flow Charting The Components Of Teaching Strategy

William R. Arnold

The same principles can be applied to problems that are assigned. That is, the teacher should assign problems that can promote desired learning. He should include easy and difficult problems in the assignment and should let each pupil elect to solve some, but not all, of the assigned problems. Quite often the difference between a dull or an exciting lesson depends on the examples, exercises, and problems utilized.

There is increased demand for the teacher to consider environments other than the classroom for learning mathematics. Children are going to resource centers, computers, and laboratories. The teacher should attempt to select an environment which will facilitate learning. Since teaching situations vary immensely, this should be done by attempting to make appropriate use of all available resources. For example, the teaching of conventional procedures for computation can be accomplished in the classroom, while finding a solution to a problem might require the children to go outside to take measurements. Increasingly, teaching machines can perform expository functions, thereby freeing the teacher for discussion and exploration moves. There are numerous materials and textbooks available and their use should be determined by existing needs.

There is no magic place to begin a flow-chart lesson plan. A teacher might decide to vary his routine by going to a mathematics laboratory. He might then consider the equipment available, think of an idea the equipment can be used to promote, and outline a sequence of moves to make or pose a problem for solution. At other times a flow-chart lesson might begin when a teacher consolidates the ideas of several pages of a textbook, considers the examples to be used, and selects moves to be made in the classroom. The essence of flow-charting lessons is that the teacher builds daily teaching strategies which enable him to vary the routine and are suited to intended learning. Several lessons devoted to rote learning and drill can be made relevant by moving to a related laboratory experience. Other lessons dealing with problem solving can be used to point out the need for learning some more routine procedures.

To begin flow-charting lessons the teacher should select an idea in mathematics with which he is familiar and write it down in terms of what the child should do. Then he should think of several possible sequences of moves and write down one that seems to have success potential. During this time the teacher should account for the examples, exercises or problems, and materials to be used. Good initial results are achieved if the lesson is simple and occurs in the classroom. As the teacher gains proficiency, more complex lessons can be planned and taught. With a little practice and study most pre-service and in-service teachers can become strategists who systematically vary their teaching style, expose children to all of the mathematics curriculum allocated to them, and make the instruction appropriate to learning.

Flow Charting The Components Of Teaching Strategy

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Flow Chart Lesson Plan (Generalized form)

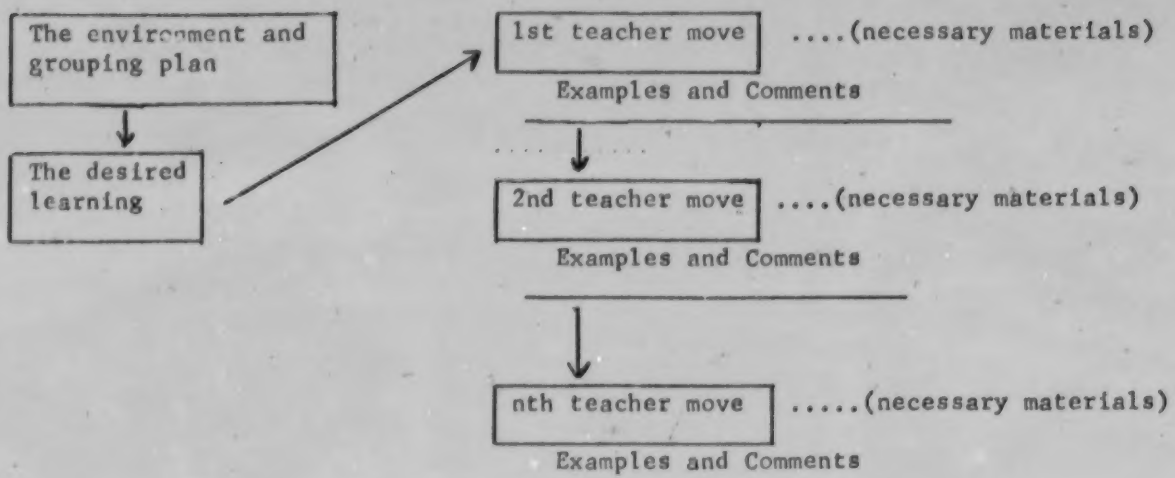

Figure 6.1 Flow-chart lesson plan, generalized form. (Note: In a complete plan there may be several desired things to learn. These would be listed on the left with arrows leading to appropriate moves or with new moves noted.)

The flow-chart lesson plans which follow were developed and successfully taught by pre-service and in-service teachers. They reflect some desirable differences in style. The intent in presenting them is to let the reader see how aspects of a teaching strategy are composed to form a unique plan for teaching.

The essence of Flow-Chart Lesson Plan No. 1 is that the teacher uses a pattern illustrated on the board so that the children discover negative integers. He decides that letting the children explore the pattern and then discuss possibilities would best accomplish his objectives. Perhaps some of the pupils' comments will include: "It can't be done"; "Minus one"; "one less"; and so on. If any feasible answer is given, the teacher writes down — 1. The discussion is pursued to get at the subtractions 5 - 7, 5 - 8....and the solutions −2, −3.... If the discussion is successful, the teacher will explain the numbers −1, −2, −3... have the names "negative one," "negative two," "negative three," and so on, because it is unlikely the child can discover these conventional names. The implication in the lesson is that discussing patterns which have been explored can be a powerful aid to discovery. This lesson was taught successfully to a group of first-grade children late in the school year.

Flow Charting The Components
Of Teaching Strategy

William R. Arnold

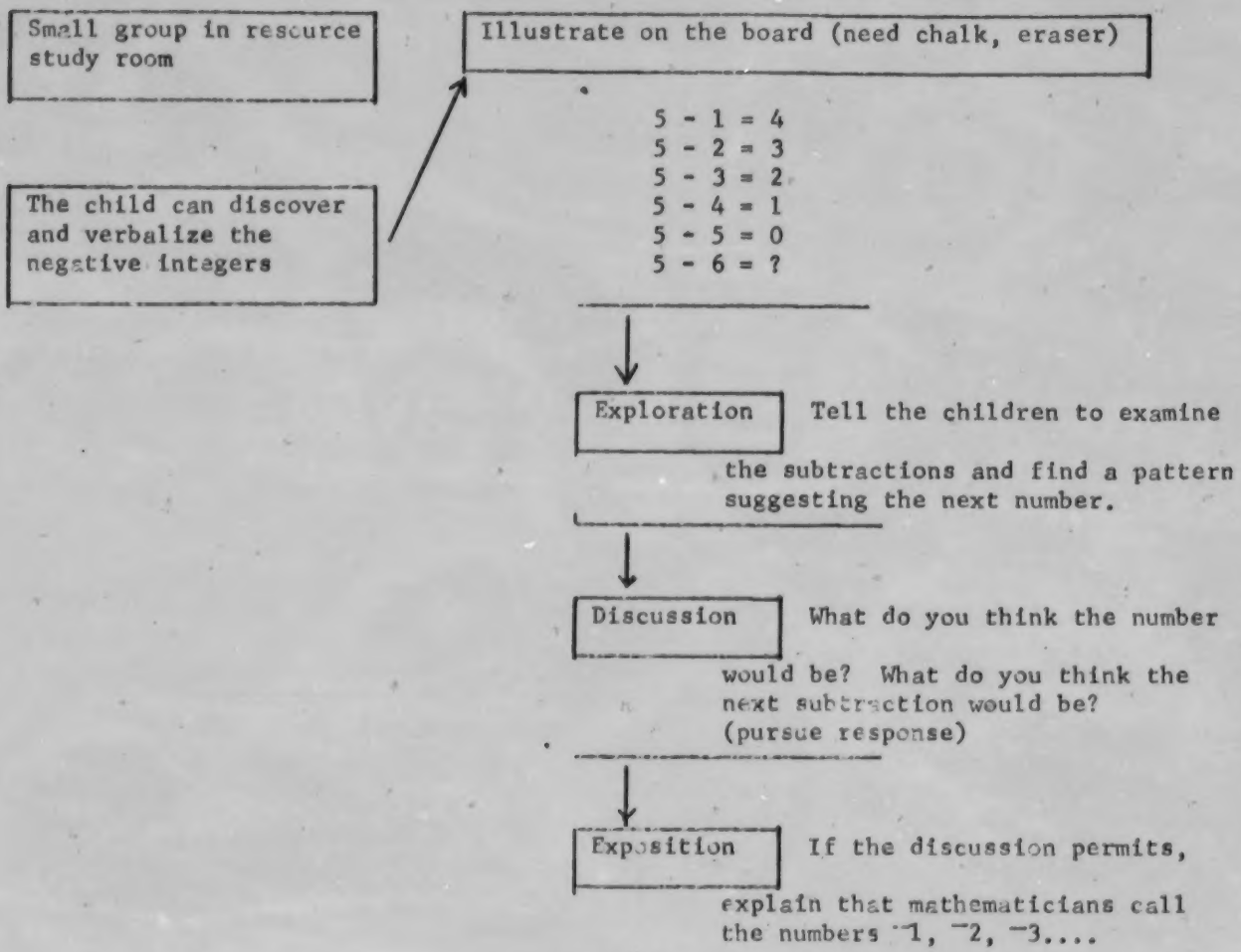
Small group in resource study room
Illustrate on the board (need chalk, eraser)
5 - 1 = 4
5 - 2 = 3
5 - 3 = 2
5 - 4 = 1
5 - 5 = 0
5 - 6 = ?
The child can discover and verbalize the negative integers
Exploration
Tell the children to examine the subtractions and find a pattern suggesting the next number.
Discussion
What do you think the number would be? What do you think the next subtraction would be? (pursue response)
Exposition
If the discussion permits, explain that mathematicians call the numbers ⁻1, ⁻2, ⁻3....

Figure 6.2 Flow-chart lesson plan no. 1

Not every idea in mathematics is subject to discovery. For example, the algorithms which facilitate computation are, for the most part, conventionalized procedures to be learned. The Flow-Chart Lesson Plan No. 2 deals with an intermediate algorithm for multiplication.

The primary intent of this lesson is to give the child a tool which will facilitate computation. Therefore, the teacher proceeds in a straightforward manner. Since part of the lesson deals with practice and providing individual help, the assignment given to the children plays a major role.

The focus of Flow-Chart Lesson Plan No. 3 is on a problem. The teacher wants the children to use knowledge and skills previously learned with as little help as possible. He is attempting to minimize the number of moves he makes, and maximize the role of the child as director of his own actions. This is a type of lesson

Flow Charting The Components
Of Teaching Strategy

William R. Arnold

which should be developed often in order to maintain interest and to make the mathematics program meaningful.

Through the use of flow-chart lesson plans the teacher can bring a degree of precision to planning which permits him to develop sequences of related lessons where he is at times active, at times passive, occasionally uses a textbook, sometimes uses the mathematics laboratory, and in general is meeting pupil needs.

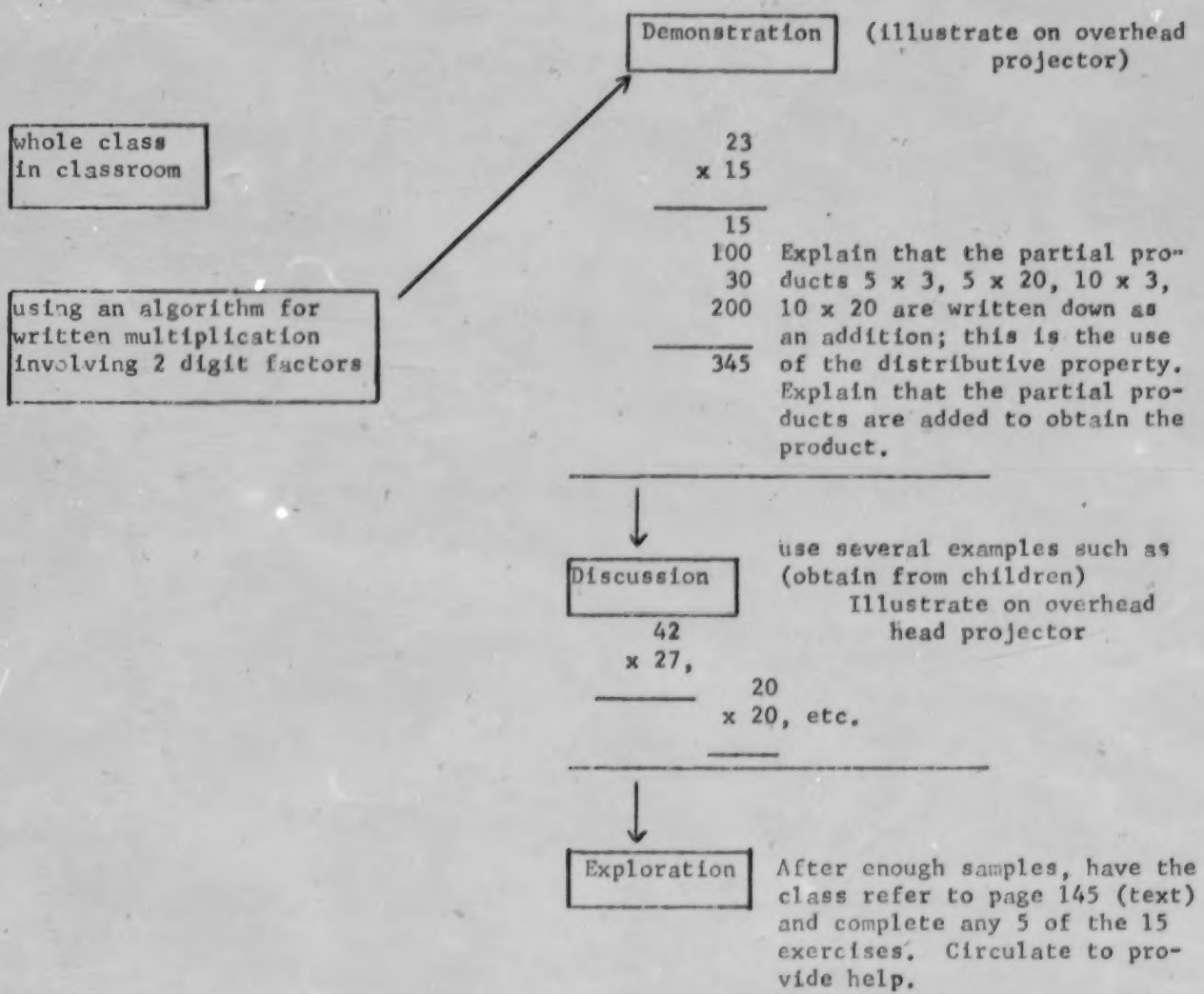
Demonstration
(illustrate on overhead projector)
whole class in classroom
using an algorithm for written multiplication involving 2 digit factors
23
x 15
15
100
30
200
345
Explain that the partial products 5 x 3, 5 x 20, 10 x 3, 10 x 20 are written down as an addition; this is the use of the distributive property. Explain that the partial products are added to obtain the product.
Discussion
use several examples such as (obtain from children)
Illustrate on overhead head projector
42
x 27,
20
x 20, etc.
Exploration
After enough samples, have the class refer to page 145 (text) and complete any 5 of the 15 exercises. Circulate to provide help.

Figure 6.3 Flow-chart lesson plan no. 2

Flow Charting The Components
Of Teaching Strategy

William R. Arnold

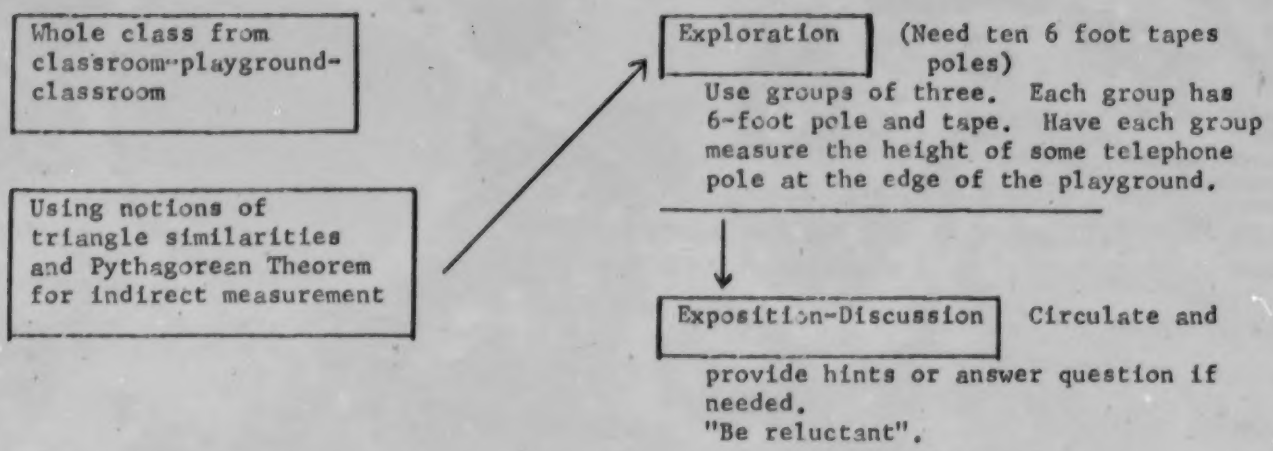
Whole class from classroom-playground-classroom
Using notions of triangle similarities and Pythagorean Theorem for indirect measurement
Exploration
(Need ten 6 foot tapes poles)
Use groups of three. Each group has 6-foot pole and tape. Have each group measure the height of some telephone pole at the edge of the playground.
Exposition-Discussion
Circulate and provide hints or answer question if needed.
"Be reluctant".

Figure 6.4 Flow-chart lesson plan no. 3

The pre-service and in-service teacher should be able to work out the task of teaching a mathematics lesson, develop a flow-chart lesson plan written in a form suggested by the general model, and subsequently use the plan to teach the lesson. When the lesson has been taught, the teacher can and should assess the effectiveness of the particular teaching strategy, and either proceed to develop new lessons or enlarge on the lesson already taught.

The Neglected Receiver: The Child

There is great need to probe into the conditions of student life and the confrontations of identity, self-esteem, and sense of personal worth and potency which enter into the student's readiness to be a receiver of educational opportunities. This writing will probe this area and mobilize some data which will help us to identify with the situation of the young ones on whose collaboration we have to depend if meaningful learning transactions are to occur. Getting children and teachers to play together in the same orchestra pit is a different job from getting scientists to play together. At the present time, educators and children are playing in quite different orchestras, by and large.

Six phases of curriculum development and change that bridge from the intellectual activity of the producer or integrator of knowledge to the learning activities of the child.[1] The phases are identified as: (1) the process of discovery, creation, formulation, or integration of basic knowledge; (2) the conversion of this knowledge into curriculum resource units, materials, frameworks, etc.; (3) a double phase - distribution of new resources by linking agents (such as publishers, regional laboratories, ESEA Title III projects, and schools of education) and the search for them by teachers, curriculum directors and, in a few interesting cases, groups of parents; (4) the adoption or adaptation of the new resources; (5) presentation of new resources to the learners by teachers in classrooms; and (6) consumption and utilization of new resources by learners. We are concerned here only with the last step in this process, the activity of consumption or learning by the students. Observations are organized under five questions:

1. How do we neglect the learner?
2. What is the learner's existential learning situation as he sits in the classroom?
3. How does the learner cope with our educational efforts?
4. What are the conditions of an effective learning situation?
5. How do we train a learner to become a good learner?

HOW DO WE NEGLECT THE LEARNER?:

There are a number of ways in which we tend to neglect the learner in our teaching efforts or our preparation of materials for teaching efforts. First, our major focus of enthusiasm and commitment is to our own specialized area of knowledge. The concepts and units of knowledge that we want to communicate are the ones that are important to us as experts in a particular area of knowledge. We are concerned that our discipline or specialty gets a significant amount of the child's ear, eye, and think time.

Second, our communication is often, sometimes necessarily, focused on gatekeepers rather than learners. What adaptations of our materials will be acceptable to curriculum decision-makers and teachers who must be reached? We are not pressed to give equal attention to the needs and readiness of the children.

Third, our conceptions of the consumer-learner and of his learning needs tend to be abstract, global, and occupationally oriented: or, production- rather than consumption-oriented. We think of the learner as an age-graded or social-background-classified cluster or group. We generalize about "the fourth grader," "the eighth

1 Ronald Lippitt, "Processes of Curriculum Change," in Robert R. Leeper, ed., Curriculum Change: Direction and Process (Washing, D.C.: Association for Supervision and Curriculum Development, NEA, 1966), pp. 43-59

grader," "the high school senior," "the central city child," " the surburban child," etc. We also make the error of thinking of the learner as a separate entity, an individual student interacting with a teacher. This is of course an abstraction from the reality in which the learner is a member of several social contexts with many simultaneous loyalties, many commitments, and many directions of interest at any one moment. We tend to think of him in the future in producer-occupational roles like ourselves, rather than someone who now and in the future will spend most of his time as a consumer who is an intelligent or unintelligent user of the resources and services produced by others.

Fourth, we tend to ignore the young in our curriculum thinking by assuming a lack of readiness and motivation on their part to be involved in educational planning and choosing--by assuming a lack of adequate maturity and perspective to make significant contributions to the selection of educational goals, methods, resources, or groupings.

Anyone who has taught children, field-tested materials, or closely observed the educational scene in other ways can probably add to this list.

WHAT IS THE EXISTENTIAL LEARNING SITUATION?:

What is the actual life-space of a young learner?

We must always remember that each day for our student is a medley of interventions, aimed at influencing him in some way--interventions by parents, teachers, peers, friends, neighbors, TV. He or she is very busy responding, reacting, initiating contacts, seeking responses.

If we focus on the classroom group, as a group, we find a number of crucial factors that interact with the teacher's efforts to provide learning opportunities and to stimulate interest in curriculum resources. In a large proportion of the classrooms in our studies, over half the children believe that the majority of other classmates disapprove of active collaboration with the teacher, of reciting and showing enthusiasm for classroom projects, of asking the teacher for help, and of being friendly with him. Actually, in a significant proportion of these classrooms, the majority of the children privately feel otherwise, but they take their clues from the active, antilearning, more hostile and impulsive members of the classroom group.

In most classrooms, the distribution of interpersonal acceptance and power among children is skewed like the curve of distribution of wealth in the nation, or more so. A few students have most of the positive acceptance of their peers and most of the power (in terms of opinion and influence leadership), whereas most of the students have very little. This state of deprivation focuses the energy of most students on interpersonal status issues and away from academic learning challenges. Those with deprivation of interpersonal status are found to be underutilizers of their intellectual capacities in comparison with high-accepted children, as measured by discrepancies between IQ and achievement.

When we study the basis of peer status in the classroom group, we discover that being perceived as good at school work is <u>way down the list</u> of valued criteria. Other skills and resources are valued much more highly.

The Neglected Receiver: The Child

When we observe the interaction of teachers with students, we find the teacher's behavior reinforcing the peer interactions. Low-status children get far more negative feedback from the teacher than do high-status children (generally, all children get more negative than positive feedback), and the children reciprocate with negative alienation from the teacher. The findings show, impressively, that the more negative the child feels toward the teacher the less the teacher can influence him to accept learning opportunities.

When we study the child's self-concept, or self-worth feelings, we find that most frequently he perceives teachers, and other adults as seeing him more critically than positively, and as being inaccessible to influence by him in terms of having his needs heard. Negative self-conception is highly related to low motivation to learn.

Although the child's occupational role from first grade for many years is that of learner, few young ones have a meaningful conception of and commitment to the role of learner, or a concern about the skills of competence as a learner. There is no meaningful or active involvement in learning goals; no vision of one's self as a learner; no sense of collaboration with the teacher in defining and clarifying learning goals, nor of collaboration with peers in learning. For example, in one study the most frequent meaning of "helping each other in the classroom" was "cheating."

Each classroom is also a deprived learning enviroment because the teacher is isolated, separated from the innovations of his colleagues and from opportunities to explore and adopt and adapt the resources available to the school system from the larger educational world. Like students, teachers lack the sophistication, values, motivations, and skills to seek out and incorporate the vast medley of available resources. This does not mean that they are not ready to work on these challenges. But they need help.

Evidence concerning the role of parents in education indicates that there is a tremendous lack of parental collaboration; that there is little orientation toward such collaboration; that parents resent teachers; and that parents apply pressure on their children in relation to school achievement in very unskilled ways.

Still other data show a marked lack of congruence and communication between school systems and the five or six other major socialization programs of the community.

The findings just reviewed display a sample of the confused, competing, deprived learning situation of the learner-consumer in the classroom and in the larger educational community.

We turn now to our third question.

HOW DOES THE LEARNER COPE WITH OUR EDUCATIONAL EFFORTS?:

How does the student cope with the variety of efforts to influence him to get him to listen, to get him to learn something? To understand our consumer's behavior at this point we need to use the social-psychological concept of overlapping situations. The child, as he sits in the classroom, exists simultaneously in more than one psychological and social field, each of which is exerting influence on him. We have already identified such fields as the teacher, peer-friends, peer-group, parents, and older-peer models. There are others. In a typical study not long ago, teenagers identified from 10 to 40 persons and groups influencing their decisions and behavior

in a week. At any one moment the child or the youth probably cannot report clearly all the competing inputs, but they do exist in different degrees of potency and awareness. What does he do to manage these often competing, confusing inputs? We have identified a number of means that students use to solve this problem. Four of these problem-solving postures are offered as illustrations.

One posture is what might be called the compartmentalized solution. This could also be called situational opportunism. Whatever stimulation is dominant at the moment is heeded at that time and other influences are blocked out or not utilized as resources. The teacher's presentation of curriculum opportunities will be attended to only when it is situationally more salient than peer relations or other interests. There is obviously little development of personal decision-making identity and internal commitment to learning with situational opportunism as a problem-solving posture.

A second problem-solving posture is what we call the pervasive dominant loyalty. This method of simplifying the problem is to make one reference group or one figure the dominant loyalty and behavioral guide across all situations--blocking out, rejecting, or withdrawing from other competing voices. This posture reduces the discomfort of decision-making. It also inhibits the development of the student as a self-initiating inquirer, learner, or decision-maker, and often results in a conforming dependency on parents, on a teacher, or on a peer group.

Yet another problem-solving posture is that of striking a balance, sometimes called a "computer solution." This is a posture of trying to please everyone as much as possible--finding compromises, doing school work but also meeting the demands of friends, trying to do projects but also following the clues of the popular kids. The amount of commitment to learning depends on the balance of forces between all the confrontations active at any one time.

Another familiar solution has been called a-plague-on-all-your-houses-- the rejection of all external authority pressures. This posture is used frequently by drop-outs and delinquents, as well as by others. The irritation and confusion of the conflicting and competing demands for the young person's energy made by parents, teachers, and others is reacted to by behavior which seems to say, "If you can't agree then I am free to do whatever seems attractive to me at the moment--my impulses are my best guide." This withdrawal from the influence arena receives support from the child's need for autonomy, from his negative feelings about authority, and from the attractiveness of the inner pleasure-seeking voices.

Even when the teacher does successfully attract or compel the focal energy of the child or the youth his learning behavior usually leaves much to be desired. What has he learned to define as successful learning effort and outcomes? Most frequently, the learning effort which is made and rewarded might be called a lip-service response or cognitive closure--something that can be intellectually categorized and fed back to the teacher in some form. There is very little sense of relation to here-and-now living, to emotional commitment, to value confrontation, or to action challenge. The child may learn to be an observer or analyst, but not an actor or volunteer. He may develop skills as a critic but not as a creator. He may become sophisticated but not committed. He may become a very sensitive responder to clues and cues but not an initiator of questions for action.

While creative educators today are focusing more than ever on the problems which I have outlined, our total educational effort is still one in which motivation is

discouragingly low and learning opportunities are squandered. It is certainly one of the most tragic resource wastes in our culture.

WHAT ARE THE CONDITIONS OF AN EFFECTIVE LEARNING SITUATION?:

What are some conditions of an effective learning situation that can be derived from the preceding findings? All of us want children to take advantage of more learning opportunities and to be presented with better opportunities. The derivations I make from the conditions that I have described are the following:

1. The teacher must involve the learners in the development of learning goals and learning norms. Group norms must support rather than inhibit learning; they must involve the students in the development and maintenance of learning ~~conditions.~~

Let us take an example from a fifth-grade class which will illustrate how these norms can be developed. The class has been asked to respond to this confrontation: "If a visitor from Mars walked in here today who was interested in education but knew nothing about American education, what would be the things that, if he saw them, would be signs that we were having a good learning day, and what would be the signs that we were having a bad learning day?" A steering commitee the members of which change each week has been set up. The first steering committee represents four of the sociometric power figures of the class--two pro-learning and two anti-learning leaders. They appoint the next steering committee, which appoints the next one, and so on, the rule being that everyone goes through steering committee membership before anyone is on a second time. By the time the less influential children become members of the steering committee, group norms about how the process operates have been established.

The first task of the steering committee is to provide leadership in establishing criteria for good and bad learning days. They post the criteria on two large sheets of newsprint. The teacher, of course, has helped to train the steering committee to lead a problem-solving discussion and to avoid, for example, being punitive toward others who express "queer" ideas. The teacher contributes her own criteria and is active in stimulating the search for resources and ideas in addition to those the children first put up.

Each day, during the last fifteen minutes, a member of the steering committee reports his observations with respect to the two sets of criteria concerning how the class has done that day. There may also be discussion and decisions about changing the criteria before the next day begins. The next day a different member of the steering committee is observer, and at the end of the week, the whole steering committee leads a half-hour discussion on whether they want to revise the criteria before they turn the material over to the next steering committee.

This example is but one way in which learners can become involved in setting their own goals and norms.

2. The second condition is that the teacher must accept the reality of the peer power structure. He must accept that there are others in the classroom group who in many ways have more power than he does about the learning process and he must involve these peer leaders in improving the learning conditions of the classroom.

The Neglected Receiver: The Child

I have mentioned above how peer power was used in the first steering committee. As another example, I think of a seventh-grade teacher of a bi-racial, lower-class classroom in which four out of five of the influential power figures in the group (three of them on probation) were antischool, antiteacher leaders. After thinking carefully about the risks, the teacher convened them for a lunch period, supplying sandwiches, and in essence said: "You fellows know, of course, that you are the ones on the questionnaire last week who the other kids said were most listened to in this class group and so you are leaders. Now the Board of Education and the principal have said that I am supposed to be leader. What are the things that we leaders can do to get together to make things better for the rest?" At the end of three sessions of this kind there was a complete change in the learning culture of that classroom, attained through support of the class leaders for norms of helping with learning procedures.

3. A third condition is that we must work to change the distribution of acceptance and influence in the classroom. I am reminded, for example, of a sixth-grade classroom that I recently visited. At the back of the room was a directory of "who is good at what in our class." If anyone needed help, whether it was shooting baskets in the gym, or working on math, or whatever it was, they used the directory to find out who would be a good tutor, a good helper for their particular need. The teacher was developing a norm similar to that taught in one of our elementary behavioral science laboratory units. There the focus is on how the group can develop norms of acceptance and support for varied ability and for recognition of the differences in developmental level of people of the same age in the same group at the same time. The unit contains many examples of how older students can be used to help in the resources development of the younger.

4. The teacher must open up channels of communication between himself and students and among students in order to identify and work on the problems which block learning. Not long ago, I was in a classroom where the children were doing ratings of the way in which they perceived the teacher and her help to them in learning, and the teacher was rating them as a learning group. After making the ratings, they put the two sets of data on the board so they could see how the teacher rated them and how she was rated by them. Then they discussed who needed to change in what ways to make the class a better place in which to learn. There would be, I think, a great upsurge in the quality of education and the motivation of students if the school staff acted as though there were a National Association of Students for the Improvement of Education with an active chapter in their building. This would make a great difference in the psychological orientation of teachers.

5. The teacher must do everything feasible to adapt the learning opportunities and tasks to the differential needs, capacities, and interests of the individual pupils in the class. I have mentioned the resource directory. There are many other ways of developing the norm of peers helping peers in the classroom.

Bronfenbrenner's research in Russia showed a very extensive development of the responsibility of children who have different levels of competence in the same classroom group to help each other. There is a variety of helper plans. One of the crucial elements often left out in most countries is that there must be peer training in the techniques of helping and of giving help.

One of the most exciting projects that we have been engaged in is that of older students acting as academic helpers of the younger students. Right now, for example, in a central city school, we have a high school where 30 volunteers act as academic

aides in the junior high and elementary schools. There are also about 30 junior high school students working in the elementary school and about 30 fifth- and sixth-graders teaching in the first and second grades. The high school helpers are in a behavioral science course for which they get regular credit. They get three periods of field work a week, being teachers and helpers, and two periods of seminar work on understanding the motivation of young ones and the techniques of helping. One of the best results is that the academic achievement and commitment of the older students increases as the individualization of instruction for the younger students increases.

A great problem in this individualization and inquiry planning is just how to facilitate the search and retrieval of materials that the inquiry model requires. As Herbert Thelen has pointed out, older students could become the consultants and helpers in retrieval activities for younger students in operating inquiry laboratories. The teacher certainly must do all that is possible to help the children understand and use conceptual models to focus their search for knowledge with which to meet inquiry confrontations. We have found that third- fourth-, and fifth-graders get tremendous excitement out of using circular process models of causation and concepts of resource development.

6. Everything possible must be done to increase the congruence and continuity of the child's relations to his various teachers and to the other socialization agents and to increase his own active role in decision-making in reciprocal influence relations with them. The learner must become an active selector among the medley of opportunities, and he must become an intelligent defender of his personal integrity and selfhood against the many demands which to him are meaningless, or which demand dependence, or which are inappropriate to his own level of readiness and maturity. He must learn to be an intelligent defender against all the encroachments on his time and energy.

7. Everything possible must be done to increase the school's environmental support for innovation, for the sharing of innovations, and for motivation to search for and use new practices. We have had some very exciting times in school systems with nomination procedures for identifying creative practices across buildings and across classrooms, and with developing documentation interviews to get concrete descriptions of the practices which make it easy to transmit them.

We have developed panels to rate which of the many identified innovations are worthy of visibility and should be spread. Such procedures for the sharing of practices legitimize what is typically regarded as rather illegitimate. Inventive teachers usually avoid sharing, saying to themselves, "Who am I to be blowing my own horn?" and "They would raise their eyebrows at me." The teachers who should be seeking resources from others believe the administration would think that "I am weak if I ask for help."

8. Finally, everything possible must be done to involve parents in the learning objectives of their children and to achieve congruence and collaboration between teachers and parents in providing support for learning, including support by the older siblings. There are many interesting ways to brief parents on what would be helpful support, such as the telephone conference. In our work with teachers, we sometimes have eight or nine teachers in different locations in telephone conferences at the same time. The same procedure would be quite easy in the briefing of parents. An electronic secretary could be used, so that parents could call a telephone number and record any information they have, to be picked up at any time during the day by the teacher. The development of room meetings as genuine work sessions rather than as "show-off" periods is also important.

The Neglected Receiver: The Child

But no matter how well we provide better conditions for learning, one of the most exciting and neglected jobs is the direct consumer education of the child.

THE EDUCATION OF THE CHILD TO BE A BETTER LEARNER:

The child must learn to seek, use, and create better educational opportunities. It seems clear that we want and need learners who are active seekers-initiators rather than passive recipients.

The children, although they are not organized nationally, do have tremendous collusive power to enforce slow-downs and sit-in techniques. We want consumers who are selective, who can use criteria of relevance rather than being uncritical buyers of what is offered. We want consumers who are eager for new learning tastes, eager for new risks rather than conforming and self-satisfied with having gotten such-and-such a score on a test. We want consumers who can reward themselves for good decision-making and for significant action efforts, rather than for getting the answers right for the teacher. With these criteria in mind, let us review briefly what we know about motivation and learning by means of the following types of training for learning. I have seen all of these approaches used, and they seem to be a critical part of the educational enterprise.

1. Training in the attitudes and skills of learning from adults, of having creative, active, cross-generational communication. We have in our social studies curriculum, for example, a unit on "learning from grownups," in which the students actually study behavioral specimens, which are role-played, of relations between teachers and students. The students in the roles soliloquize: What is going on? Why they are not asking for help? Why are they asking for help all the time and being overdependent? The other pupils observe and analyze in lab teams and come up with their own proposals of how these situations might be changed and how communication between teachers and pupils might be improved. Then the teams put on demonstrations of improvements, giving their evidence as to what would cause a difference in learning if a different outlook on learning from adults were developed.

2. A second training focus is on how to use resource persons other than the teacher. In our training classes, a resource person often observes students putting on the board questions they really want to explore, which demonstrates where they are in their thinking. The resource person has a chance to listen. The bottom third of the board is left empty. The resource person is then asked to add his expert questions which help get at the kinds of information the students want to know. Then we move into a dialogue of the group with the resource person. The group often develops an interview schedule to use with resource informants.

3. Another area of learner education must be learning the skills and responsibilities of reciprocal influence. Instead of using the slow-down and sit-in which are negative resistance postures and procedures, the students must learn positive, responsible methods of influence. These skills are frequently developed through mutual feedback and evaluation of what is going on in the classroom. The teacher reports ways in which he feels the learner role is being played inadequately and the pupils give their thoughts about ways in which they are not getting the kind of help they would like.

In our "bridging the generations" project, we have had, for several years, lower-class and middle-class teenagers working together on the problems of communication with adults. After studying the problem of communication in their seminar they decided

to initiate teacher- and parent-education programs to get the parents and teachers involved in understanding the way teenagers look at the world. So, with consultation, they developed teacher and parent nights in which they would typically present a role-playing episode of a problem of communication between teenagers and parents, or between teenagers and teachers. Teachers, parents, and teenagers would sit in diagnostic teams, observing this episode and collecting data on "What do we feel are the reasons for the problem of communication here?" "What do we think would be ways that it could be changed to improve it?" They would move from these diagnostic teams into demonstrations of how communication might be facilitated.

Teachers and parents do not refuse invitations to participate in these groups. They may not accept the invitation to come to a PTA meeting, but they will not turn down the invitation of teenagers to come to a teenage-parent or teenage-teacher session when the teenagers are the initiators.

4. Children and teenagers must be helped to link their learning intentions or commitments to learning actions and efforts. When the citizens of one city that we are working in became aware of the lack of correlation between knowledge of right and wrong and actual behavior patterns of youth, they formed a curriculum development committee made up of a rabbi, a priest, a Protestant minister, a General Motors executive, the associate superintendent of schools, and two social scientist helpers to work on an educational program which is now called The Deciders. They worked on a curriculum in value education and decision-making that would be acceptable to the values experts but would be nonsectarian enough to be accepted by the school system. They came out with a very exciting program--a "hit" in training materials.

The training groups usually were led by pairs of citizens, sometimes a teacher and a layman, sometimes a husband and wife. They worked on the problem of linking what we know is the thing to do to actually doing it. Typical self-inquiry projects that were generated were equilibrium analyses of the problems, "What are the forces for and against putting more learning effort into a committed learning task?" and "What are the forces for and against my doing any of the things I have just planned to do?" One group member said, "This is the first time in my life that I have been honest."

5. We need to help the young ones learn to use peers as learning resources. The lack of communication is tragic when help-seeking and help-giving are not norms of practice. I have mentioned a classroom helper directory. In most situations, we need to start with methodological training on skills of helping.

6. Identification with good teachers, or modeling, must be supplemented by practice and support for practice. Learning to teach others is another one of the important learner-training techniques that I have mentioned. The great response of failing lower-class black students in becoming active learners as they study the problems of young ones like themselves and the kind of help they need is inspiring. They are really studying themselves at a safe distance, by studying others like themselves who are younger-looking at their own problems of motivation and ego needs. They talk very freely in sixth-grade and junior high seminars about ego needs, about overcoming resistance, about being a good model, and about not creating dependency.

7. There must be training to function in problem-solving inquiry teams. There are lots of notions about the need for inquiry teams, but very little training in how to be a good team. Students often experience serious failures; if the teacher has to step in frequently, dependency is fostered. Therefore, training in skills

to make learners productive in group and team situations is essential, no matter what the curriculum content.

8. Finally, there is need for training to develop internalized bases of self-reward and the feelings of success. Children can get great excitement from working on the questions: "What is the meaning of learning for me?" and "What are the signs that I am learning something?"

CONCLUSION:

All this may sound a bit idealistic or impractical in terms of expectations for teachers, but my observation is that such a program of learner sensitivity training is crucial. It is the basic linkage between the curriculum and the learner. It is the determiner of the educational payoff.

Our experience is that teachers can be taught to do the job, if they are given the opportunity to learn. They, too, have been neglected. An amazing number of teachers are eager learners when they have the opportunity to experience the potential for self-renewal in personal and professional improvement opportunities. This is particularly true when they see that professional improvement can release them from energy-depleting burdens of caretaking activities for half-hearted pupils and permit them to move toward the exhilirating role of designer of inquiry opportunities--a teacher of how to learn, a presenter of exciting problem confrontations, a guide to learning resources.

ON IMPROVING ONE'S ABILITY TO HELP CHILDREN LEARN MATHEMATICS*

Phares G. O'Daffer

INTRODUCTION

Have you noticed the increasing number of teachers who prefer to say that they are "helping children learn mathematics" rather than "teaching mathematics"? What one says, of course, does not always describe accurately what one does; but this change in emphasis has had, and can have, a profound effect on the approach a teacher takes toward children and mathematics.

The focus on how children learn mathematics has given rise to new attempts to create a more appropriate learning environment for children. The resulting innovations, from Individually Prescribed Instruction to the mathematics laboratory, have been extensively discussed and often used. Yet many teachers who are well aware of these recent innovations still experience difficulty making them work in their classrooms. They sympathize with the child's need for freedom to learn, but they have made only limited improvements in their ability to create conditions that allow the child to exercise this freedom.

The suggestions that follow are based on the assumption that the improvement of one's ability to help children learn mathematics is an endeavor that begins in small ways and evolves as a result of a continuing process of teacher self-renewal. These ideas will, we hope, help you devise new ways of adapting aspects of recent innovations to your current style of mathematics instruction. Perhaps they will also serve as a catalyst for an ongoing program designed to improve your ability to help children learn mathematics.

LEARN YOUR A-B-C's

To effectively help children learn mathematics, you need to know children, to know mathematics, and to find workable methods for involving one with the other. This requirement, rather obvious but rarely satisfied, is implied and elaborated on in the abbreviated "ABC's of mathematics learning" presented in figure 1. A study and extension of this figure may help you place the phenomenon of mathematics learning in perspective.

If you conscientiously attempt to keep up with the key new ideas in each of the three components of mathematics learning described in figure 1, you will have taken an initial step toward improved competency in the classroom.

The questions that follow, along with references such as those suggested and listed at the end of this article, may serve to keep your quest for information in focus:

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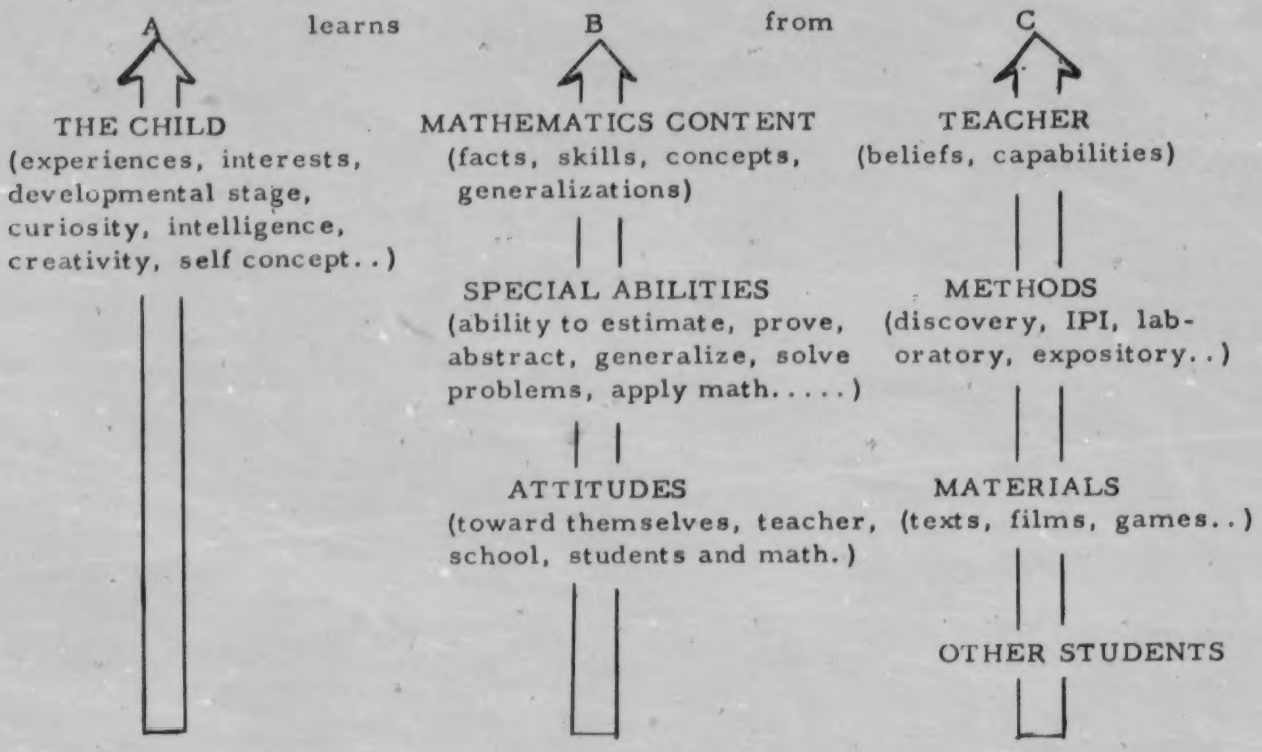
*The Arithmetic Teacher, November, 1972; pp. 519-526.



NOTE: In the above relation, the letters A, B, and C may be thought of as variables. The words in heavy type below each variable may be considered, in a broad sense, as naming replacements for these variables.

Figure 1.

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1. Are there new insights into the ways in which children learn? (All references; especially Biggs and MacLean 1969; Copeland 1970; and Rosskopf, Steffe, and Taback 1971).

2. What mathematical ideas have recently been suggested as appropriate for children? (All references; especially Biggs and MacLean 1969; Dienes 1960; and Dienes 1964).

3. Which materials and methods have been judged most useful in facilitating mathematics learning? (All references, especially Biggs and MacLean 1969; Last chapter of Copeland 1970; first two chapters of Dienes 1964; Nuffield Mathematics Project 1967; and Kidd, Meyers, and Cilley 1970).

As you search for answers to these questions, remember that you can become a better teacher by continuing, so to speak, to learn your "ABC's"

TAG THE THINGS YOU TEACH

As you consider more specific aspects of mathematics learning, especially those related to the less structured instructional settings such as the mathematics

laboratory, you may feel a strong need for a "teacher's classification" of the general types of learning involved.

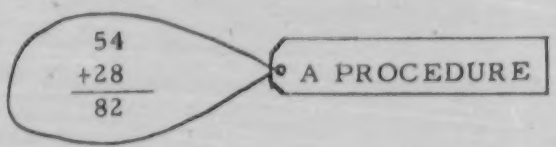
Once you have broadly categorized the types of things that children are likely to be learning, your goals will become clearer, you will feel more secure about allowing children greater freedom, and your general ability to select appropriate learning sctivities will be enhanced.

One oversimplified, but useful, categorization of the types of things children learn is illustrated by the questions, answers, and tags shown in figure 2.

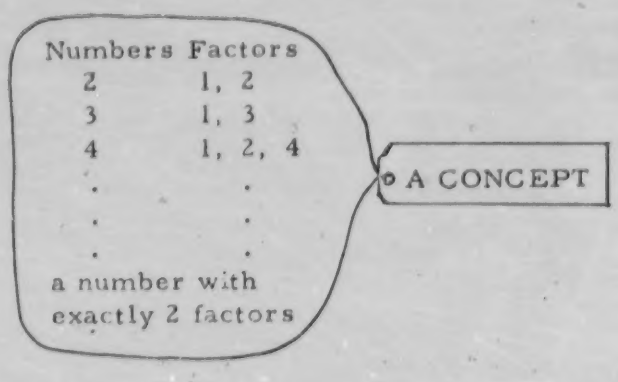
It is important to recognize that each of these types of learning have unique characteristics. Because of this, the approaches and the children's activities that are chosen to promote these learnings may often be quite different.

1. Do you remember what 9 x 6 equals?

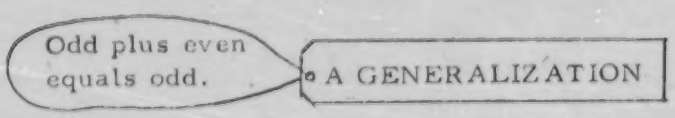
2. How do you do this addition?



3. Do you understand what is meant by "prime number"?



4. What can you say about every addition where one addend is odd and the other is even?



5. How do you feel about solving problems in mathematics

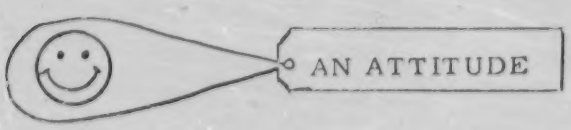


Figure 2.

For facts (such as 9 x 6 = 54), it is helpful if the child is able to recall them quickly when they are needed. Hence, the learning process might include an activity or a game that requires the child to practice recalling such facts.

For procedures (such as adding 28 to 54), it is recognized that an efficient technique (a speed skill) may be meaningfully evolved, with careful nurturing, from a more understandable, but longer, technique (a power skill). Thus, the learning process might start with the method illustrated here;

$$\begin{array}{r} 54 \\ +28 \\ \hline 12 \\ 70 \\ \hline 82 \end{array}$$

and gradually seek and practice shortcuts until the desired level of efficiency is attained. For other higher-level procedures (such as estimation, proof, and problem solving), it is often not useful to think in terms of an evolution from power to speed.

For concepts (such as prime number), it is desirable for the child to be able to recognize examples, give examples, and describe concept characteristics. The learning process might build the foundation for this by involving the child in different physical activities that embody the concept. For example, the child can be asked to list those numbers that tell the number of squares in rectangles that can be cut from graph paper (if no single rows of graph paper squares may be cut).

For generalizations (such as The sum of an odd number and an even number is an odd number), the child may need to use specific instances in solving problems. Hence, the learning process might include opportunities for the child to discover the generalization from a sequence of carefully chosen instances, or for him to produce additional instances of the discovered generalization.

Where attitudes are concerned (such as the child's level of confidence in his ability to cope with mathematical problems), it is recognized that the child must experience success, and that having some fun with mathematics may decrease the fear that erodes his confidence. So the learning process might include the creation in the classroom of a "fun with mathematics corner" that contains such things as puzzles, mazes, and design materials in a deliberate effort to encourage the child to successfully play with mathematics.

By tagging the things you teach and considering the learning environments necessary to successfully teach them, you may find a new role in your classroom for some of the recent innovations in methods or for some previously omitted ideas of mathematics.

PROCEED WITH A PLAN

So you are learning the ABC's and you have useful tags on the things to be taught, now how do you proceed in the daily task of helping children learn mathematics?

An outline (inherently flexible) around which daily learning experiences may be planned could be a valuable organizational aid for you, and it could give you a fresh insight into the role of new approaches to instruction.

Here is an outline that has proved to be quite useful, especially for those teachers who have wanted to make a modest beginning toward allowing children more opportunities for active involvement with mathematical materials and ideas:

1. Preparation
2. Investigation
3. Discussion
4. Utilization
5. Extension

The investigation phase is fundamental to any learning experience. In this phase, the children are encouraged to become actively involved, individually or in groups, in the investigation of a situation that contains the seed for the principal ideas of the lesson. Homemade or commercially produced manipulative materials often provide the setting for the situation to be investigated. At other times, simple teacher-devised activities provide this stimulus. For example, the activity card shown in figure 3 might have been made by a teacher to initiate an investigation in

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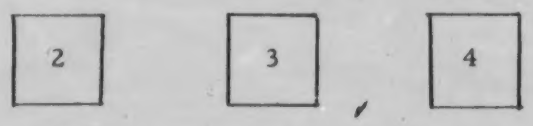
Label three cards like the ones shown here.



Turn the cards over and mix them up.

Pick any two cards. Multiply the numbers on the two cards.

Multiply the product that you got by the number on the third card.

Do this at least five times.

WHAT CAN YOU SAY ABOUT THE FINAL PRODUCTS?

Figure 3.

a lesson designed to help children form this generalization: You can arrange three factors any way you please and the product will always be the same. The activity indicated in figure 4 uses sets of colored rods and might be used in the early development of the concept of even number.

To set the stage for the investigation, a preliminary preparation phase is often needed. This phase provides for a brief review of key ideas that are needed for the investigation, and for any motivational activity that would be helpful in introducing the investigation.

Following the investigation, a discussion phase allows teacher and child to share ideas on what they found in the investigation. In this phase, the teacher has an excellent opportunity to ask questions and supply examples that will help children

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How many rods or trains can you find that can be matched by a train of two rods of the same length?

(Record the numbers for the rods or trains you found.)

Figure 4

to further develop their understanding of the embryonic ideas germinated in the investigation. Children are also encouraged to discuss investigations with other children as another means of shaping their ideas.

The discussion phase might be considered preparatory to the utilization phase. This phase consists primarily of an independent practice experience for the child in which he utilizes the ideas learned in the investigation and discussion phases. This experience, often called "seat work" by teachers, might involve practice recalling facts; practice using developed skills; exercises where generalizations are applied; or additional individual experiences designed to broaden the child's understanding of a concept or to help him use his problem-solving ability. For example, the exercises shown in figure 5, which are to be completed independently by the child, might be a part of the utilization phase following the discussion of the investigation phase suggested in figure 3.

Finally, the extension phase provides for the use of remedial or enrichment activities to further individualize the learning opportunities.

If you choose to plan your lessons as described here, you may find that the investigation provides a modest increase in the amount of active student involvement with mathematics materials. If however, you choose to maximize the investigation phase while de-emphasizing the others, then you might quite correctly be accused of using the laboratory approach. On the other hand, if you maximize the discussion phase,

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Use only the information in the box to find the products.

7 x 11 = 77	17 x 91 = 1547	13 x 11 x 7 = _____
13 x 77 = 1001	13 x 11 = 143	17 x 13 x 7 = _____
13 x 7 = 91	143 x 17 = 2431	11 x 17 x 13 = _____

Figure 5

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then you may find more options for a guided-discovery approach to mathematics learning. Also, it is possible that maximization of the utilization phase, accompanied by appropriate student materials, would allow you to embark on a course of individually-prescribed instruction.

Regardless of the blend of investigation, discussion, utilization, and extension that you choose for a given learning experience, your consideration of these phases in your instructional planning may well have benefits for the students in your classes.

RESTRICT YOUR RECITATION

In the realm of the classroom, it has been suggested that a new adage replace the old, and that the teacher, rather than the child, should be seen but not heard. Perhaps this is a bit extreme, but a conscious effort by the teacher to talk less, and involve children more, will have a valuable payoff.

In a good investigation, it is not uncommon to see children deeply involved and assuming full responsibility for completing the task at hand. The teacher, who played a key role in initiating the investigation, may appear to be not needed as he moves about the room. Occasionally a child asks about procedure, and he willingly offers a clarification. Another child asks for an explanation or verification of his tentative results, and the teacher responds with a question that returns the responsibility for evaluating, checking or rethinking back to the child. Clearly, a spirit of learning exists here that could easily be dissipated by too many teacher words. As he observes the children in the process of investigation, the teacher might well be most effective by remaining somewhat mysteriously non-committal on child-or teacher-posed questions about the ideas involved.

The discussion phase, as described earlier, follows the investigation and provides for a discussion of the child's findings and for a more systematic development of the ideas. Even in this phase, it would be well for the teacher, as much as possible, to play the quiet role of possessor of knowledge, rather than the more verbal role

of professor of knowledge.

In a good discussion, it is not unusual for children, when they have reached an impasse in their thinking and communication about an idea, to ask the teacher if he can clarify the point. This is when the teacher emerges as a resource person. At other times, when ideas that are new to the teacher arise, the teacher participates in the discussion as a co-learner with the child. Both situations are conducive to a comfortable, meaningful discussion, but both can easily be disrupted by a teacher who talks too much. The more teachers talk, the more decisions they are making for the children. And for eager youngsters, nothing is more stifling than for someone to have removed all possibilities for them to decide anything on their own.

Perhaps by restricting your recitation, by "watching your words," you will be able to stimulate, rather than stifle, the children in your charge.

QUESTION YOUR QUESTIONS

If the way to limit the need for words is to get children involved with ideas, then perhaps we should seek better ways to elicit this involvement.

One path to success lies in improving your art of questioning. What better way to minimize the need for words than to substitute an appropriate question for a wordy explanation at a time when the child's interest in a mathematical situation is beginning to ripen?

For example, suppose a group of children understand the concept triangle and are ready to consider characteristics that distinguish one type of triangle from another. To initiate the investigation, an appropriate question might be this:

> On a five-by-five geoboard (see figure 6), you can form triangles by placing rubber bands around the nails. How many different triangles can you form that have no nails inside? Record each one that you find on a piece of graph paper.

This question is an activity-stimulating and activity-sustaining type of question.

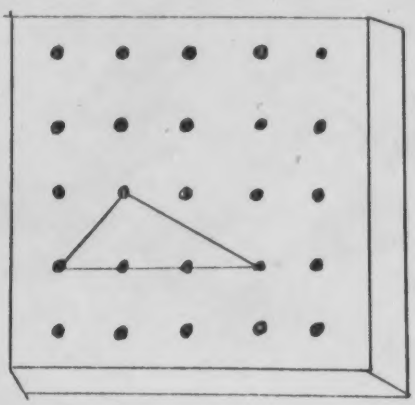

Figure 6

It helps get the child involved in a search that he will continue with a modicum of further motivation. Notice, also, that the answer is not as important as the experiences the child has as he responds to the question. Further, the question is clear enough so that the child immediately becomes involved with the challenge of the investigation; he wastes no energy trying to make sense out of the question.

Another special characteristic of this type of question is that it has built-in provisions for individual differences. Specifically, when you ask a question of the how-many-can-you-find type, the child can feel successful even if he finds only one. (For the geoboard question, the more persistent child may find as many as twenty-eight different triangles, if a duplication of a triangle in a different position is not counted.)

Of course, not all investigations can, or should, be introduced by this type of question. However, you may want to try to find several good questions of the how-many type that have the potential for involving children in an investigation of mathematical ideas.

Questions also play an important role in the discussion that follows the investigation. Since it is the child who is involved in the investigation, it seems reasonable that the child's ideas about the findings would be quite valuable. By listening to the child and asking appropriate questions, the teacher can build on the child's initial ideas to help him develop a deeper understanding. This understanding cannot be developed, however, by questions that can be answered correctly by simply recalling a fact or performing a practical skill. Nor is it sufficient to ask questions that require a decision by the child, but for which a guess, or yes or no answer will suffice. Rather, we should look at the tags on the things we teach and ask questions that are appropriate for facilitating each particular type of learning.

For example, the geoboard investigation might set the stage for the development of the concepts of isosceles triangle, right triangle, scalene triangle, and equilateral triangle. In the discussion phase of this learning experience, the teacher might ask questions such as the following:

1. Can you choose a pair of the triangles that you found and describe ways in which one is different from the other?

2. In what ways are some of the triangles that you found alike? (Children may respond with statements like "some have square corners," "some have two sides the same," "some have no sides the same," "some are large," and so on.)

3. How would you describe a triangle that is different from any of the triangles you formed on the geoboard?

Questions such as these encourage children to focus on the common properties of the triangles (such as "two sides the same") in a certain set of triangles, thus helping them to form a concept (such as the concept of isosceles triangle). These are questions that are specifically conceived to help children develop concepts (as opposed to questions that help the child review facts), learn procedures, discover a generalization, or even alter an attitude.

What kind of questions do you ask? If you question your use of questions, the answers might help you find questions that will play a new role in your classroom.

REMEMBER THE IMPORTANT THINGS

Knowledge of children, mathematics, and pedagogy, valuable as it is, is like dry tinder for a fire. A spark is needed to ignite it. This spark appears as a result of a certain style of teacher interaction with both the child and mathematics. This type of interaction, very difficult to describe or measure, seems to be a manifestation of the teacher's basic beliefs about children, about mathematics, and about how to help children learn mathematics.

Pedagogical beliefs have been discussed and questioned in earlier paragraphs, but what are your basic beliefs about children? Do you respect for them as unique persons with special interests and abilities, sensitive feelings, and important thoughts? How do you feel about their need for a natural, free atmosphere in which they can explore, make decisions, be independent, and try exciting new things? Are you sensitive to their simultaneous need for a supportive, nonjudgmental atmosphere in which they have enough routine activities to provide a comfortable threshold of security? What importance do you attach to their need for success and for the accompanying sense of worthiness? What else do you believe?

What are your basic beliefs about mathematics? To what degree do you appreciate the unique historical contributions of men and women to the development of mathematics? Do you view this development as complete, or do you see mathematics as an ever-growing discipline with new vistas opening up and with new problems to solve? Do you feel the beauty of mathematics and view it as a possible source of pleasure? How do you view the impact mathematics has had as a result of its application to other disciplines? Are you personally curious about certain aspects of mathematics and, at your level, sometimes involved in the exploration of interesting ideas? Are you confident and enthusiastic in approaching mathematical situations with your children? What else do you believe?

In essence, if a teacher desires to create a new climate of interaction with children and mathematics that will spark effective learning experiences, he must realize that it usually comes through significant experiences that provide for teacher self-renewal--a revision of basic beliefs. Clearly, and we are back to the ABC's again, it is through searching for more insight--reading, studying, observing, experiencing, experimenting, rethinking--that one's basic beliefs are altered.

CONCLUSION

It is a rare teacher who knows children, who knows mathematics, who is skillful at involving children with mathematics, and who makes use of his knowledge of the types of learning involved. Exceptional, too, is the teacher who has a well-conceived plan, and who substitutes effective questions for excessive talk. Still more unusual is the teacher who adds to all this tinder the spark of quality interaction with children and mathematics, thus creating an environment aflame with learning.

A quest toward improving your ability to help children learn mathematics may not

lead you to this rare combination of competencies--but then again, it might.

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"Characteristics of Learning Experiences to Facilitate Creative Behavior" *

A. Activities BEFORE Lesson (Reading, art, music, social studies, science, etc.):
1. Confrontation with ambiguities and uncertainties precipitated.
2. Anticipation and expectation heightened.
3. Awareness of problem to be solved, difficulty to be faced, gap in information to be filled, etc. heightened.
4. Building only existing information or skills facilitated.
5. Concern about a problem heightened.
6. Curiosity and wanting to know stimulated.
7. Familiar made strange or strange made familiar by analogy.
8. Freedom from inhibiting sets facilitated.
9. Looking at same material from several different psychological or sociological viewpoints promoted.
10. Provocative questions requiring learner to examine information in a different way and in greater depth asked.
11. Predictions from limited information required.
12. Purposefulness of activity made clear.
13. Assignment structured only enough to give clues and direction.
14. Taking the next step beyond that what is known encouraged.
15. Warm-up provided in some way (easy to difficult, familiar to unfamiliar, bodily involvement, etc.).

B. Activities DURING Lesson:
1. Awareness of problems and difficulties heightened during progress of lesson.
2. Creative and constructive rather than cynical acceptance of limitations in information, skills, etc. facilitated.
3. Creative personality characteristics or predispositions (willingness to attempt the difficult, freedom to conform or nonconform, etc.) encouraged.
4. Creative problem-solving process replicated in stories, accounts of history and invention, etc.
5. Creative processes described, illustrated, and illuminated in stories, historical accounts, etc.
6. Exploration made deliberate and systematic.
7. Incompleteness of information presented.
8. Juxtaposition of apparently irrelevant elements precipitated.
9. Mysteries explored and examined.
10. Open-endedness preserved.
11. Outcomes not completely predictable.
12. Predictions from limited information required.
13. Reading with imagination (to make it sound like the thing happening) encouraged.
14. Search for truth facilitated by honesty and realism of materials.
15. Skills for finding out identified and encouraged.
16. Surprises heightened and deliberately used.
17. Visualization encouraged.

C. Activities AFTER Lesson:
1. Ambiguities played with.
2. Awareness of problem, difficulty, gap in information, etc. facilitated.
3. Awareness and acknowledgment of teachers of pupil potentialities based on response.
4. Concern about problem heightened.
5. Constuctive response called for (other ways, better ways, etc.).
6. Continuity with previously learned skills, information, etc. facilitated.

"Characteristics of Learning Experiences to Facilitate Creative Behavior" *

7. Constructive rather than cynical acceptance of limitations encouraged.
8. Digging more deeply required, going beyond the obvious
9. Divergent thinking made legitimate.
10. Elaborating Upon what is read encouraged.
11. Elegant solutions (simpliest solutions taking into account largest number of variables) encouraged.
12. Empathetic metaphor to give new feeling or understanding of object, person, or state facilitated.
13. Experimentation required.
14. Familiar made strange or strange made familiar by analogy.
15. Fantasies examined to find solutions to realistic problems.
16. Future projection encouraged.
17. Going beyond text materials encouraged.
18. Improbabilities entertained.
19. Irrelevance (apparently) accepted and used.
20. Judgment deferred until pool of ideas has been produced.
21. Knowledge from one field brought to bear on another.
22. Looking at same material from several different viewpoints encouraged.
23. Manipulation of ideas, objects, information encouraged.
24. Multiple hypotheses encouraged.
25. One thing permitted to lead to another.
26. Paradoxes confronted and examined.
27. Play in pushing a fundamental law to its limit encouraged.
28. Possible causes and consequences called for.
29. Provocative questions used.
30. Potentialities discovered and tested.
31. Reorganization of information required.
32. Returning to previously acquired skill, information, etc. to see new relationships encouraged.
33. Self-initiated learning encouraged.
34. Skills for finding out practiced.
35. Synthesis of different and apparently irrelevant elements facilitated.
36. Systematic testing of hypothese encouraged.
37. Taking next step beyond what is known facilitated.
38. Testing and revision of predictions provided.
39. Transformation and rearrangement of materials encouraged.

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* Extracted from: E. Paul Torrance:
ENCOURAGING CREATIVITY IN THE CLASSROOM
Wm. C. Brown Co., 1970

GAMES CHILDREN PLAY....TO LEARN

Learning's not all books. Sometimes it's games. And here are some of the best, how they're used and what they teach.

Take four plastic cubes. Color them tricky--green, red, white and blue on various sides. Now match them up so that each cube shows a different color.

Sound familiar? If so, you may be one of the millions of people who paid one dollar last year for a disease that comes wrapped in plastic. It's called Instant Insanity. If you've ever tried it, you know why.

What does Instant Insanity have to do with games in the classroom? Just this: Anyone who can figure out how four hunks of plastic can unfidget fidgety kids and make sober adults toss aside responsibilities knows what a game is. More important, he knows what a good game is--and, probably, what a good classroom game is.

GRADE TEACHER recently had a long talk with the man who made a success of driving people crazy, Instant Insanity creator Franz O. Armbruster. We contacted other people too--game designers, toymakers, producers of educational materials.

Our purpose: To find out what the professional "gamesmen" are doing for your classroom. To get ideas that would help you select and/or create better games for your pupils. And to help you use the games you choose more effectively.

If you're like most teachers, you've already discovered the magic of games in the classroom. You know what they can do to motivate, reinforce and enrich a lesson.

If that's so, then you must know what a good classroom game is. Or do you?

Talk to the experts and you may not be so sure. By their standards, some of your old favorites would sink.

Take the spelling bee. Not many people would dare to disparage that time-honored classroom tradition. Yet, as far as most of the people we talked to are concerned, it's a rotten game.

The trouble with the spelling bee is that the outcome is always predictable. Think about it: Isn't it always the same "smart" youngster--the one who can spell antidisestablishmentarianism without blinking an eye--who is left standing at the end? The kids who really need spelling practice are out of the game the very first time around.

Besides, the spelling bee isn't really much fun--for the early dropouts, it's no fun at all. As anyone who knows his games will tell you, whether a game is designed for mass market or classroom crutch, its success depends on its being fun--for everyone.

The loss of the spelling bee--or for that matter, any other game that has outlived

its usefulness in the classroom--doesn't really matter. The professional game-maker is just waiting to fill the gaps for you.

Not that he's always ready to admit it. A number of people we talked to intimated that to put the "educational" label on a game intended for the general market is to give it the kiss of death. This, despite the fact that outfits like Creative Playthings, Playskool, Responsive Environments and even Milton Bradley, primarily a mass-market distributor, have long since proved that one way to win races is to hitch your wagon to teacher's gold stars.

Still, whether it's out in the open or not, the commercial producers seem determined to provide prepackaged history, science, logic, math, spelling--everything--all gift-wrapped in fun and laughs for your pupils (for some specifics, see the list at the end of this article). With the market place bulging with wares, the problem becomes one of sorting those that are right for your classroom from those that just won't work.

Something Going For You

On that score, you've got something going for you--your own knowledge of your pupils needs. Alice Gordon of Abt Associates, Cambridge, Mass., who developed a whole history program around the use of games for Science Research Associates, put it this way: "How good a classroom game is depends on how well it achieves its purpose. That, only the classroom teacher can decide."

Mrs. Gordon is dead set against the use of games as classroom busy work. Most of the others we contacted seemed to agree.

They also tend to go along with most of the guidelines for game selection offered by Instant Insanity man Franz Armbruster, though he describes them as "personal criteria biased by all kinds of prejudices":

1. Is the game fun? Says Mr. Armbruster: "If it isn't fun, forget it. You might as well do the regular classroom work."

2. Are the elements of the game really those of the subject you're trying to teach? If the lesson is on Spanish phrases, a game about life in Madrid or Barcelona isn't going to help, unless it uses Spanish phrases.

3. Is the game behavior specific? In other words, does the scoring and winning depend on skill rather than luck?

4. Is the game self-perpetuating? Can a child, once taught the game, teach it to his peers? (Nobody needs to tell you how important that is in the classroom.)

5. Does even the loser benefit in some way--preferably, by improvement in the game skill?

Winners and Losers

It's on this last point--the idea of winners and losers--that there was some disagreement among the people we spoke to.

Herbert Sprigle, director of the Learning to Learn School, Jacksonville, Fla., who developed SRA's "Inquisitive Games" series feels competition should be played down in the classroom. But that's probably because his interest is the preschool, where building a good self-image is all-important.

One of those taking the other view is Al Rice of Schoolhouse Visuals, Inc., Silver Spring, Md. Competition figures in his company's new game, Action Fractions, and he feels that's as it should be. Says he: "Adult life is full of competition, and the school is supposed to prepare children for adult life."

Mr. Armbruster, who now heads his own company, Products of the Behavioral Sciences, in Campbell, Calif., shrugs the whole thing off with: "It's the adults who make a fuss. Kids thrive on competition."

The same rules that apply to testing commercial games for classroom ability should apply to those created by that little old game-maker--you. And, judging by the mail we get at GRADE TEACHER, chances are you're no piker at the game game yourself.

For teacher-gamemakers, Mr. Armbruster offers this suggestion: "Develop the games around the things that appeal to children. Then they'll enjoy the game, and learn something from it."

According to him, the best way to find out what children like is this: "Take two or three of your kids to the dimestore and tell them to get lost. Wherever you find them after 10 or 15 minutes is where you should buy the makings for games."

Some sure winners with children: a lot of small objects of the same kind--bottle caps, beans or buttons, for example; rope, wire and other hardware department items; water; playing cards. (Mr. Armbruster's firm recommends a deck in which the cards have standard backs and blank faces, so that today's game can be wiped away to make way for tomorrow's .)

"Kids love to break things, too," Mr. Armbruster says. "Someday, I'm going to develop a game that lets kids pop, crash and smash to their hearts content."

Color, Mr. Armbruster maintains, is only important to a game if it has something to do with the play. Monopoly, in his opinion, represents good use of color (colors help identify property blocks), while the standard deck of cards represents a poor one. "Each suit should be a different color," he says, "or all suits should be the same."

Missing a Bet

It doesn't matter whether the games your children play in the classroom are those you make yourself, or those produced commercially. But if you don't try them at all--then you may be missing a bet.

As Ruth Patmore of Wff'n Proof Publishers points out, it's a whole new way of teaching. She puts it this way: "You're not standing at the chalkboard trying to get and keep the attention of a whole class. You're mixing. You're arbitrating disputes. And you're helping the children resolve problems they pose for each other."

"Today's student wants greater participation in the learning process, and in the planning which that process requires. Games," she says, "are designed for today's teacher, too."

The Marketplace

If you need more convincing on the game question, take a look at the following list. It's a small but representative showcase of the many items on the market that are designed for, or could be adapted to use in the classroom.

Language Arts: If you're looking for replacements for the old fashioned spelling bee, you shouldn't have any trouble. Among the possibles: Word-Roll, a new release for beginning readers by Pictorial Publishers (distributor: McCormick-Mathers); Play 'n Talk by Parker Brothers; Go Gin, by Ideal; Tumble Words, by Kohner Brothers.

At a more advanced level, for building word power, there's Avalon Hill's Word Power. Designed for the mass market, it's got plenty in it for the classroom, too, including a booklet that goes into such things as the origins, definitions and romance of words.

Reading: Pic-A-Pair is a reading and word study program made up of games that start at the pre-school level and go on through sixth grade. It's produced by Titan.

From Lyons and Carnahan, there is the Phonics We Use Learning Games Kit for pre-schoolers and beginning readers. The games are based on formats similar to those on the popular market.

Also for beginning readers is "Get Set," a series of eight games, which represents Houghton-Mifflin's biggest venture into the educational games field to date, and a wide range of magnetic puzzle kits by American Toy and Furniture Co.

For more advanced readers, there's Avalon Hill's Shakespeare. The game can be played three ways: the basic rules require no prior knowledge of Shakespeare and his works; the other two versions get progressively more dependent on Shakespearean knowledge.

Faculty Cards--it's produced by Aco games--teaches phonic word construction, word recognition and vocabulary. It can be used at a variety of grade levels and with mixed groups, because point values are determined by the ages of the players.

Social Studies: New this year from Cadaco, Inc. is See the USA. The game board is a map decorated with scenic views.

Conestoga, by Broman Percepta, is an historically based game, simulating the hazards faced by pioneers on the Oregon Trail. Such games as Gettysburg, D-Day and Afrika Korps, by Avalon Hill, are also true to historic fact.

Math: Big thing in Math games is the egg timer. Tuf, an equations game distributed by the company of the same name, for example, uses three of them.

Among other math possibilities: Fracto, by Broman Percepta, now being tested in schools; the "Inquisitive Games" set, Exploring Numbers and Space, by SRA which is designed for the early education classroom; and Arithmequick, by Design Enterprizes.

Thinking Skills: Wff'n Proof's "Games for Thinkers" are as much cult as games. Wff'n Proof clubs have been organized in schools, children put out Wff'n Proof newsletters and there are even Wff'n Proof Olympics staged between children from various parts of the country. New games are always being added to the line, and while they deal with various disciplines, they all are essentially thinking games.

Robert B. Davis

Whether one thinks of the aeronautical engineer versus the pilot, or of the pure mathematician versus the engineer, or of the economist versus the businessman, virtually every major field of human endeavor is split between a group of theorists and a group of practitioners. It seems to me that it is a sign of health when the two groups are able to communicate, and when they attempt to work closely together. Speaking as one of the clinical people or practitioners, I want to thank the theorists for their efforts at bridging the gap which sometimes seems to separate us.

My remarks are made from the practitioner's point of view, which is the only one I can legitimately claim. Based on the Madison Project work of the past eight years, I want to cite a few examples and a few ideas related to them.

Specifically, I have three goals; first, I want to give examples of what we have regarded as discovery experience for children, in the hope that some of you will be able to suggest a few of the ways in which these experiences differ from exposition, and even differ from one another. This last remark deserves emphasizing, for it is my present notion that there are many different kinds of discovery experience, and we confuse the issue badly when we treat discovery as a single well-defined kind of experience.

My second goal is to offer a few remarks attempting to interpret or describe some of the things we mean by discovery, and to explain why some of us believe in its importance.

Finally, I shall list a few things that might be called objectives of Madison Project teaching, for this, too, may clarify why we think we believe in 'discovery.'

SOME EXAMPLES AND SOME INTERPRETATIONS

The students in my first example were some low-IQ culturally deprived children in the seventh grade, some of whom were older than normal for grade seven. The topic was the matter of finding pairs of whole numbers that would satisfy linear equations of the form

$$(\_\_ \times 3) + 2 =$$

Although we found that $\_\_$, notation useful, and consequently did use it with these childlen, what I have written above could be translated into traditional x,y notation as

$$y = 3x + 2$$

We passed out graph paper and suggested that suitable pairs of whole numbers be recorded according to the usual Cartesian use of coordinates.

Very nearly all of the students made the obvious discovery that there is a simple linear pattern to the resulting dots. A considerable number of them actually applied this discovery, by extrapolating according to the pattern, then checking to see if their new points gave numbers that satisfied the equation.

It is these students who used the discovery this way whom I wish to discuss. Some of them spent several days working with various linear equations, using the patterns to find points, then checking by substituting into the equation. This was one of the earliest things the project

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* From Learning by Discovery: A Critical Appraisal, Lee S. Shulman, Evan R. Keislar, eds. (Chicago: Rand McNally, 1966), 114-128.

ever did that seemed to fascinate children far beyond their normal degree of involvement with school.

My conjecture is that the important aspect of this was perhaps a combination of achievement in making their own discovery, competitive gratification vis-a-vis those classmates who did not make the discovery, autonomy of having set the task for themselves, some intrinsic esthetic or closure reward, and the existence of a verification that did not depend upon the teacher.

These were culturally deprived children, with a middle-class teacher and a middle-class curriculum. Previous observation had already convinced us that these children were at best tolerent of a schoolish learning that was wrong in every important respect. The children were always checking up, between school and social realities outside of school, and invariably found the school to be wanting.

The school taught you to speak the language incorrectly--for example, the school taught, 'It is I' in a world where the social reality was, 'Its me'--the school taught personal economics incorrectly, the school taught civics incorrectly, and so on. Nothing ever checked when you tried it out against actual reality. (For example, if a majority of people in the county don't like the sheriff, he won't be reelected--according to what they teach in school. But in reality, perhaps a majority of the people are not even allowed to vote--to use an example that is in our newspapers at the present moment.)

Here, in their own discovery, they had found something that really worked! If you tried it out, it checked out perfectly!

In any event, some of the children were captivated, and spent several days on the matter, asking to take graph paper home and so on.

(Incidentally, in the long run--i.e., over the year that we worked with this class--truancy decreased markedly, and parents reported their children taking an unprecendented interest in school--for example, by discussing it at mealtime.)

My second example is perhaps somewhat similar. We wanted to give children in grades 3-9 some experience with variables and with the arithmetic of signed numbers. It is one of our principles that such experience should always be provided in a sensible mathematical contect, and (if at all possible) in a form which would permit a student to make one or more interesting discoveries. We believe that this procedure helps get the children in a frame of mind where they are always poking around looking for interesting patterns that may be lurking just beneath the surface.

I should emphasize that the discoveries in question will usually not be part of the basic purpose of the lesson, and it is not essential for a child to discover them. They are primarily a bonus for those who do discover them.

In the present instance, remember, we wanted the children to get some experience using variables, and working with the arithmetic of signed numbers. We consequently gave them quadratic equations to solve, beginning with

$$(\_\_ \times \_\_) - (5 \times \_\_) + 6 = 0$$

and gradually progressing to harder problems, such as

$$(\_\_ \times \_\_) - (20 \times \_\_) = 0$$

Now, at first the only method available to the student was, of course, trial and error. If he makes no discoveries, the student continues with this method, and gets full benefit from the basic part of the lesson; that is, he gets a great deal of experience using variables and signed

numbers, and in a situation where he does not reagrd this as drill.

But - if the student discovers the so-called coefficient rules for quadratic equations, his use of trial and error can be guided to maximum efficiency. He has discovered a secret - and one which many of his classmates don't know. They may never know!

Torpedoing

In both of the previous examples, we make use of a technique which we call 'torpedoing.' After a student has discovered what he believes is the pattern for linear equations from working with

$$(\square \times 1) + 3 = \Delta$$
$$(\square \times 1) + 5 = \Delta$$
$$(\square \times 1) + 2 = \Delta$$

and so forth - after he is confident of his mastery, we unobtrusively slip in a problem like

$$(\square \times 2) + 3 = \Delta$$

He uses his pattern, he checks - and the numbers don't work!
What shall he do?

With a little thought, he discovers there is a pattern here, also - indeed, there is a more general pattern of which he had discovered only a special case.

In a similar way, with the quadratic equations, we begin by using only unequal prime roots, so that one of the two coeffeicent rules (the product rule) is extremely obvious. Using it alone leads to easy solution of the equations, such as

$$(\square \times \square) - (5 \times \square) + 6 = \bigcirc$$
$$(\square \times \square) - (12 \times \square) + 35 = \bigcirc$$
$$(\square \times \square) - (13 \times \square) + 22 = \bigcirc$$
$$(\square \times \square) - (7 \times \square) + 10 = \bigcirc$$

Here also, once the student is really pleased with his discovery, and with the new power it has given him, we confront him - unobtrusively and unexpectedly - with a variant problem which will tend to confound his theory.

In this instance, we slip in a problem having composite roots, instead of the prime roots the student had previously dealt with.

The product rule now seems to indicate more than two roots: for example, with

$$(\square \times \square) = (9 \times \square) + 20 = \bigcirc$$

many students will say the roots are $\{2, 10, 4, 3\}$.

Trial by substitution shows that this is wrong. Again, by persevering, the student finds that there is a broader theory, of which he had found only a narrower

Discovery in the Teaching of Mathematics

Why Do We Like Torpedoing?

It may seem that what I have described is simple sadism, or How to Be One Up on Your Students Without Really Teaching. We feel that it is not so (although, unfortunately, it can be in the hands of a teacher who really is a sadist).

Why do we use this technique of torpedoing some of our students' best theories?

It is important to realize that, at the outset, we don't know. We use the technique because, intuitively, it feels right.

But, in the years while we have been using this technique, we have, of course, discussed it often, and even made some analytic attempts at constructing an abstract rationalization for it.

Perhaps we like the technique because:

1. It gives the brightest students something to work on while the others catch up on more basic work;

2. It is a friendly challenge from teacher to student, and students rise to such bait better than fresh-water fish do to flies;

3. Perhaps Piaget's processes of 'assimilation' and 'accommodation' need to be practiced, and this is where you practice them;

4. Or, to put point 3 in less technical language, perhaps the security of a friendly classroom is the best place to gain experience in fixing up theories that used to work, but somehow don't seem to work any more - the classroom is a better place to learn this than, say, the political meetings of anti-integrationists or the radical right;

5. Then, too, it is worth learning that science does not deal in absolute truth. Sufficient unto each day are the theories thereof - and an irreconcilable contradiction may be discovered tomorrow! (Or, for that matter, a better theory!) If your theories work, make the most of them - but keep a wary eye out, just the same.

6. Finally, this is one of the ways that we go about bringing history into the classroom. If one wishes to understand history, one must have some background of relevant experience. Since the history of mathematics is an unending story of trials, failures, break-throughs, temporary successes, new points of view, and so on, it is unintelligible to the person who has no background experience in trials, failures, break-throughs, temporary successes, revised points of view, and so on. Torpedoing theories in the classroom provided background experience that parellels important historical phenomena. Can you realize what the discovery of irrationals meant to the Greeks if you, yourself start out with the sophisticated viewpoint of the 20th century?

The "Crisis" Dilemma: Seeking The Unit Matrix:

This next example involves a different kind of discovery situation. In the previous examples, the discoveries were merely optional bonuses added to the meat of the lesson; moreover, students discovering any secret patterns kept the secret to them-

selves, revealing their knowledge only indirectly, by using the secrets to solve difficult problems easily and quickly.

In the present example, all eyes are focussed upon a central problem which we wish to solve, if possible - or to recognize as unsolvable, if a logical argument shows that no solution can exist. If anyone finds an answer, he will announce his discovery at once.

Specifically, we look at the system of 2-by-2 matrices and ask how this new mathematical system compares with the familiar old system of rational numbers. One question is this: is there a 2-by-2 matrix which plays a role analogous to that of the rational number zero? The answer turns out to be that there is, and it is the matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So far so good. Now - is there a matrix that plays a role analogous to the rational number one? Students invariably - and wisely, on the available evidence - guess

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

but a quick computation shows that this is not satisfactory.

Here we have the dilemma: IS there any matrix that behaves like the integer one? Has our failure to find one been a symptom of the impossibility of the task, or have we merely failed due to personal reasons, not reasons of fundamental impossibility? Is it worthwhile trying any longer? If so, how shall we proceed?

This has proved to be a consistently exciting lesson for fifth-graders, or anyone older who doesn't already know the answer.

Going Beyond the Data or the Task:

In the filmed lesson entitled, "Graphing an Ellipse," at the end of a lesson on graphing

$$x^2 + ky^2 = 25, \quad 0 \leqq k,$$

a seventh-grade student (Debbie H., according to her nametag) asks: "Why couldn't you use matrices, and make a graph for k less than zero?"

Since these students have previously used matrices to introduce complex numbers, this is an ingenious and appropriate suggestion. The teacher does not immediately respond, but another student (whose nametag reads Lex) answers: "No, you can't, because you won't be able to graph matrices." This answer is essentially correct - but both of these remarks go beyond anything the teacher had planned or anticipated.

The teacher's contribution to this is mainly a genuine appreciation of the students' contribution - but we believe this is important. Children somehow act far cleverer when their cleverness is welcome and appreciated.

Discovery in the Teaching of Mathematics

Do It Your Own Way: Kye's Arithmetic:

Somewhat similar is this example. A third-grade teacher was introducing subtraction, with borrowing and carrying:

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline \end{array}$$

She said: "You can't subtract 8 from 4, so you take 10 from the 60....."

A third-grade boy named Kye interrupted: "Oh yes you can!

$4 - 8 = 4$

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline -\ 4 \end{array}$$

and $60 - 20 = 40$

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline -\ 4 \\ 40 \\ \hline \end{array}$$

and $40 - 4 = 36$

$$\begin{array}{r} 64 \\ -\ 28 \\ \hline -\ 4 \\ 40 \\ \hline 36. \end{array}"$$

The teacher did nothing here to solicit originality, but when she was confronted with it, she listened to the student, tried to understand, and welcomed and appreciated his contribution.

This was an unusual, but actual occurrence. The more common, 'traditional' response would have been to say: "No, Kye, that's not the way you do it. Now watch carefully and I'll show you...."

Where does this traditional reflection of his contribution leave Kye? He is given the feeling that mathematics is a stupid subject that never works out the way you'd expect it would....Given enough experiences of this sort - and in a traditional situation he will be - Kye will probably transfer his interest and his energy to some other field of endeavor.

This example has always seemed to me to suggest the essence of good 'modern' teaching in mathematics, as opposed to 'traditional' teaching. In a phrase: listen to the student, and be prepared for him to suggest a better answer than any you know. The 'modern' teacher actually learns from his students!

Kye's algorithm for subtracting was an original contribution of a third-grade boy. It is in many ways the nicest algorithm for subtracting that I have ever seen - and it was invented by a boy in the third grade.

Discovery in the Teaching of Mathematics

The traditional teacher assumes from the outset that such a thing is impossible. Is it any wonder that the traditional teacher somehow never encounters such clever student behavior rarely appears before the eyes of those who don't believe in its existence.

Autonomy and Proliferation:

In introducing graphical integration and differentiation to an eighth-grade class, we proceeded as follows: We obtained a print of the PSSC film "Straight Line Kinematics" which is an expositional treatment of these topics in relation to velocity and acceleration. The students could view this film whenever they wished - and as often as they wished. They were then given a shoebox full of simple equipment - the PSSC ticker-tape equipment to take home if they wished, and were asked to devise their own experiments and work up their own data. They later performed their experiments in school, at a session which has been recorded on film.

The effort and ingenuity that some students put into this went far beyond their normal effort for 'schoolwork'.

What Questions:

In one instance, we give physical apparatus to students and ask them what questions might be worth studying about this apparatus. The point here is for the student to identify appropriate questions. An eminent mathematician, Professor McShane of the University of Virginia, has said that his favorite mathematics problem is stated in a textbook as follows:

A pile of coal catches on fire.

These seven words are the entire statement of the problem - in a mathematics book!

Now - what mathematics questions does this pose?

Leaving Things Open-Ended:

Many mathematics problems occur at a stage in the child's life when he is not yet prepared to answer them. An honest use of logic seems to compel us to leave these questions open for the time being. The alternative would be to 'answer' them on the basis of authority - but we believe this would tend to make the child think that mathematics is based upon the pronouncements of authorities. We prefer to leave the question open - after all, aren't most scientific questions open at the present time? Or, perhaps, always open?

Example: Is

an axiom or a theorem? Ultimately the child will learn that is is a theorem:

$$\square + \bigcirc = \square$$
$$\bigcirc + \bigcirc = \bigcirc$$
$$\square \times (\bigcirc + \bigcirc) = \square \times \bigcirc$$
$$(\square \times \bigcirc) + (\square \times \bigcirc) = \square \times \bigcirc$$
$$\square \times \bigcirc = \bigcirc$$

This proof, however, involves some subtle and awkward points, so that the child cannot settle the question when it first occurs to him, and we leave the question open. The child knows that if he is ever able to prove the result, he will be able to classify it as a theorem. In the meantime...Who knows which it is?

Where Does It Come From? :

The Madison Project approach to logic is, so far as we know, completely unique and unprecedented. Virtually every existing book on logic tells you what modul poenes is, tells you what the truth table is for 'and', 'or', 'if...then,' and so on.

But why? Where does all of this come from, anyhow?

Given the transitory nature of scientific knowledge, we can hardly settle for facts which are static pieces torn out of the fabric of time past, time present, and time future.

Where does all of this logic come from? How would we go about making up our own logic if we wished to do so?

In order to answer these questions-at least, according to our own view of the answer- the Madison Project proceeds like this:

First, we ask children (grades 7, 8, 9, or older) to analyze statemants of their friends, and to classify them as true or false. They realize that this is a vast oversimplification, for most ordinary statements in the ordinary world are neither true nor false; there's some truth in them, but one still has some possible doubts or reservations.

We then ask the children to focus on actual usages of 'and' and 'or', and to record as many different uses of 'and' and 'or', as they can find by means of truth tables.

Ordinary language has a great many different uses of both words. One of my favorites - we might label it 'and', in order to distinguish it from other uses, which can be labelled 'and_2,' 'and_3' and so forth - is this one:

Keep driving like that and you'll kill somebody.

A somewhat similar use of 'and' occurred prior to the 1964 election, in the radio admonition: Vote, and the choice is yours, Don't vote, and the choice is theirs. If you fail to register, you have no choice. Consider, also, the poem: "Laugh, and the world laughs with you, Weep, and you weep alone." The result of this activity is an extensive truth table with many different columns, headed 'and_1',

'and$_2$', 'and$_3$', etc., and 'or$_1$,' 'or$_2$.' 'or$_3$,' and so on. Usually some students insist on moving into a more-than-two-valued logic, in order better to reflect nuances of meaning which seem to them to be present. Once we have collected together this large truth table, we have completed stage one, which might be labelled Observing the Behavior of the Natives.

We move next to the Legislative Stage: In order to gain clarity, we agree on one single meaning for the word 'and', we pick one column in the truth table to define this meaning, and we legislate that 'and', we pick one column in the truth table to define this meaning, and we legislate that 'and' shall henceforth be used in that single sense only.

The study of logic proceeds further: after the Sociological stage, and the Legislative stage, we move on to the Abstract stage, and so forth - but the point is that these children have made up their own systems of logic. As a result, they know where logic comes from. In the same way, you understand Beethoven differently after you, yourself, have written some music of your own composition.

An Active Role, and Focussing Attention. Discovery of another sort is perhaps involved in teaching students (second-graders, say) to plot points on Cartesian coordinates. The teacher plots a few points, but the students learn more by imitation than by following a careful exposition. Learning by imitation of course involves a kind of discovery, since you must figure out how the teacher is doing it.

As David Page has pointed out, the best mathematics students have always learned by discovery - even when listening to a lecture, they are actively thinking: asking Why? Why not? How about doing it this way? Now what do you suppose he meant by that? Why can't you do it this way? and so on.

Games Using Clues. One of our most successful lessons goes like this:

Three students, working together, make up a rule. For example, Whatever number we tell them, they'll double it and add twenty. They do not tell us their rule; instead we tell them numbers. They apply their rule to each number we tell them, and tell us each answer. It's our job to guess their rule.

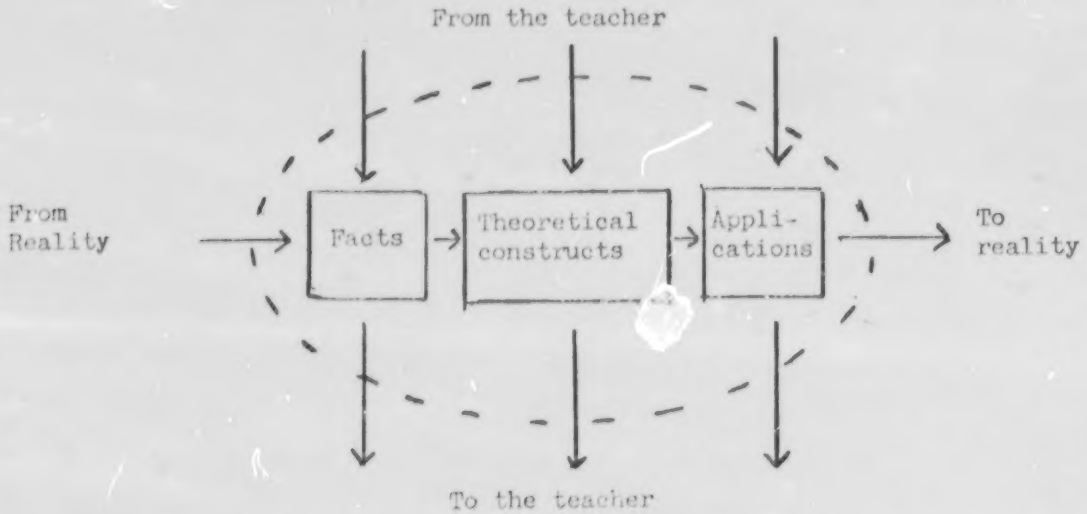
There is a great deal of mathematics involved in this lesson - for example, the distinction between formula and function, and such properties of functions as linearity, oddness, evenness, rate of growth, and so on. But perhaps the main discovery aspect resembles closely Suchman's (1964) work on inquiry: By choosing wisely the numbers we tell them, we can get clues as to the nature of the function they are using. No single clue will usually be decisive, but a suitable combination will be.

A somewhat similar format has been used by David Page in his lessons on Hidden Numbers. A few numbers are written on a piece of paper, and the students are to guess the numbers from a set of clues. To make matters harder, some of the clues are false, so that the students must recognize and make use of the contradictions which are contained in the set of clues (grades 4, 5, 6, 7, 8, 9,).

TWO THEORETICAL INTERPRETATIONS:

An extremely valuable approach to analyzing communications in class has been developed by Professor J. Richard Suchman. The data inside a student's mind at an in-

stant in time can be classified (as an oversimplification, of course) into three categories: facts, unifying mental constructs (roughly, 'theories'), and applications. The possible communications can be diagrammed as follows:



Using this analytical approach in interpreting classroom lessons can be very exciting and gratifying - which, as J. Robert Oppenheimer has remarked, is perhaps the most valid test of a theoretical approach.

Because of limitations of time and space, I shall not attempt to illustrate the use of Suchman's diagrams, beyond remarking that on a Suchman diagram, discovery communications seem to appear as conspicuously horizontal channels. Particularly in connection with the work of Piaget, Tolman, Lewin, Kohler, White, and Bruner, Suchman diagrams are a powerful analytical tool for the practitioner seeking a rationale for understanding what goes on in the classroom.

Another aspect of discovery experiences is being emphasized in the motivation studies by Richard de Charms (in press) in relation to the analytical dichotomy of the learner's perception of his role, which de Charms polarizes as 'origin' versus 'pawn'.

THE GOALS OF EDUCATION:

It is worth remembering that the life of a human being, or the interrelated lives of many human beings, are in reality unified wholes, not separated into pieces in any way whatsoever. When, in order to analyze life abstractly, we break off pieces by invoking categories, we do a violence to the reality whole that can impede comprehension as easily as it may, hopefully, facilitate it.

In particular, motivation is not separate in reality from perception, personality, learning, social interactions, or communication. Still more specifically, reality does not begin with a statement of explicit goals. If a teacher begins with an attempted statement of explicit goals, he does so by choice, and may easily do so in error, for his subsequent behavior may not reveal the same goals as his initial words did.

In our own Madison Project teaching, our original conscious goals were general

and nonspecific: we wanted to find some of the best experiences with mathematics that children could have. To choose among alternatives we relied upon our intuitive assessments.

From this highly general purpose we spun out a sequence of specific activities in the classroom. In order to discuss these experiences, after they were created, we have attempted to work out some suitable analytical categories. In particular, we have identified what appear to be some of our probable goals. Here is a tentative list, surely somewhat incomplete:

1. We want to give students experience in discovering patterns in abstract situations;
2. We want students to have experience in recognizing potentially open-ended situations, and in extending open-ended situations by original creative work;
3. We want the students to be familiar with the basic concepts of mathematics, such as variable, open sentence, truth set, function, Cartesian coordinates, mapping, isomorphism, linearity, matrices, implication, contradiction, axiom, etc.
4. We want the students to build up, in their own minds, suitable mental imagery (in the sense of Lewin, Tolman, Piaget, Leignitz, Polya, et al.) to permit them to perform mental manipulations involving the basic ideas of mathematics (such as function, linearity, isomorphism, mapping, etc., as mentioned above);
5. We want the students to acquire a modest mastery of the basic techniques of mathematics;
6. We want the students to know the basic facts of mathematics, such as $7 + 3 = 10$, $-1 \times -1 = +1$, and so on;
7. We want the students to possess considerable facility in relating the various parts of mathematics one to another — for example, using algebra as a tool in studying geometry, or recognizing the structure of the algebra of linear transformations of the plane into itself, and so on;
8. We want the students to possess an easy skill relating mathematics to the applications of mathematics in physics and elsewhere;
9. We want the students to have a real feeling for the history of mathematics, derived partly from having been eye-witness observers (or participants) on the occasion of mathematical discoveries.

We regard the preceding nine points as intellectual matters; but they must be accompanied by some emotional or value goals, namely:

10. We want the student to know that mathematics really and truly is discoverable (something few people believe);
11. We want each student, as part of the task of knowing himself, to get a realistic assessment of his own personal ability in discovering mathematics;
12. We want the students to come to value 'educated intuition' in its proper place;
13. We want the students to value abstract rational analysis in its proper place;
14. We want the students - as much as possible - to know when to persevere, and when to be flexible;
15. We want the students to have a feeling that mathematics is fun or exciting, or worthwhile.

The preceding goals do not sound like the goals of a traditional arithmetic or algebra program. They are not.

Space and time do not permit us to pursue the matter, but implicit beneath all Madison Project work is the notion that education and training are different, that

education is for people, and training is for electronic machinery (which usually doesn't need it anyhow) - indeed, all repetitious routine tasks are basically nonhuman.

One example of the distinction - a tragic and highly suggestive example - will have to serve where many might be cited:

A few years ago in a hospital nursery in Binghamton, N.Y., the formula for new born babies was made with salt instead of sugar. By the time the error was discovered, a dozen or so babies had either died or suffered severe and irreparable brain damage.

Now, - who or what was at fault? Many babies - indeed, virtually all of them - simultaneously developed feeding problems. Some of the mothers - but none of the nurses! - tasted the formula and complained that it was unusually salty.

No nurse heeded either clue. The nurses had been trained to soothe a new mother's anxieties, lest they be passed on to the infant and create feeding problems.

Those of us who have children of our own know how much this episode meant to the parents of the babies involved.

New York State has responded by passing a new law, which, I believe, makes it illegal for a hospital to store salt in a room in which nursery formulas are prepared.

This is a significant step in the wrong direction.

Every time we attempt to by-pass human resourcefulness - by laws, rote training, or otherwise - we move toward, and not away from, the unintelligent behavior of the nurses who were trained but not educated. The response of the mothers was more appropriate than that of the nurses, but the nonadaptive blind weight of authority decided the outcome in favor of the 'trained' nurses.

I think it important for every teacher always to remember that he, the teacher, does not know the right answer or the right response - he can only hope that, when the time comes, his former students will respond appropriately.

Those theorists who study education are on shaky ground indeed if their analysis assumes that they can separate right answers from wrong answers. If the matter in question is trivial, perhaps they can; but if it is important, they surely cannot.

The present emphasis on creativity and divergent thinking would never have occurred - and should never have occurred - but for the fact that we had gone all too far down the road labelled training, and had, surprisingly, lost sight of education.

DEPTH LEARNING IN ARITHMETIC—WHAT IS IT?*

/ Charlotte W. Junge

The acquisition of skills has been consistently recognized as one of the basic aims of education to which the elementary school should contribute. From colonial times to the present, much of the attention of schools and of teachers has been centered on this aspect of education. Furthermore, this preoccupation with the development of skills has been associated largely with the "three R's" Reading, writing, and arithmetic, frequently referred to as tool subjects, have been presumed to encompass all the skills which are of real importance in the education of children.

Although it is true that effective use of language and numbers does require high-level skills of a complex sort, the three R's involve far more than skills. One of the most damaging and limiting conceptions in elementary education arises from the point of view that developing competence in language and arithmetic is largely a matter of gaining command of specific skills. Curriculums and teaching procedure; based on this idea result in rote learning, the memorization of specific facts, and the teaching of "things." All forms of language and mathematics are essentially concerned with meanings. They represent ways of gaining, interpreting, and transmitting ideas, and explaining relationships. These processes rest for effectiveness on basic skill, but the controlling principle in learning is meaning and relationship.

Efficiency in habit and skill alone is not a satisfying end-point of education, if for no other reason than that life itself is not a simple problem in logistics, a problem for which any good Answer Book has the precise formula worked out to the fourth decimal place!

When learning is at its best, it will provide opportunities for the child to produce original ideas, to be active in his own learning, and to give range to his imagination. Somewhere in our teaching procedure we must provide time for contemplation, for experimentation, and for wondering. Was it not Einstein who said to us, "Those who have ceased to wonder are as good as dead"? It is possible that in our own failure to wonder we do not see the importance of opportunities for children to wonder? All too often in the pressure for efficiency, in our desire to make the curriculum more "rigorous," and in the crowding of assignments up to the pupil's capacity, we forget to save time for the learner to give free expression to the understandings he is developing: to question, to wonder, and to experiment beyond accepted patterns of operation in arithmetic. Like automatons, children are trained to record, retain, and reproduce larger and larger amounts of material to which someone else has given shape and form. Seldom do they experience the excitement of discovery, never the risk of being wrong, never a moment in which they can say, "Poor though it is, it is my own!" For these children learning in arithmetic has taken on a routine quality. It is a surface thing lacking in meaning and in depth.

It was just a few short years ago that arithmetic was the "red-haired-stepchild" in the elementary curriculum. Now we ride the crest— due to public interest and concern as well as interest and concern within the profession. For the first time we have the financial support to bring the changes in the curriculum which are necessary to meet the needs of our times and the capabilities of our students. In our headlong pursuit of excellence we must not lose sight of a very basic goal in teaching— the development of the ability to think quantitatively. It would be very easy under present pressures to make record, retain, and reproduce the three R's of arithmetic!

This is not to say that acquisition of the basic facts and principles of arithmetic is unnecessary and unimportant. It is to say that if real learning is to take

Depth Learning In Arithmetic—What Is It?*

Charlotte W. Junge

place on the part of the student, then one must view knowledge as a means, not as an end, in arithmetic education. The building of accurate, well-organized concepts is, from the point of view of the learner, a creative process in which thinking plays a leading role. All thinking is dependent upon a well-organized body of knowledge, thoroughly understood and pertinent to the problem. As we all know, "Wisdom does not come to those who gape at nature with an empty head."[1]

DEVELOPING DEPTH IN LEARNING

Depth learning in arithmetic is not developed, then, by emphasis upon skills alone. Neither is it developed solely by providing children with "enrichment" activities or with content of a more difficult nature taught at an advanced level. It is developed in quite a different way, that is:

1. by confronting the student with challenging problems—but problems within his power of comprehension,
2. by leading him in those methods of disciplined thought that have been found to facilitate the work of mathematicians,
3. by maturing him in those methods of disciplined thought that have been found to facilitate the work of mathematicians,
4. by providing opportunities to discover and to find original solutions, and
5. by steadily encouraging him to new levels of creative thinking.

Depth learning in arithmetic results from a way of teaching which encourages reflective thinking, which permits experimentation and originality, and which holds steadily before the learner the need for knowing the facts and principles necessary for clear thought. Given these things, children will move on to condepts of greater difficulty and the development of greater skill in handling quantitative relationships.

Note that I have not spoken here of abstraction, but I have spoken of precision in thinking and meaning. Thought may be precise whether abstract or concrete. Note that I have not spoken of rigor. Concepts correctly understood and precisely expressed at any level will embody a standard of rigor appropriate to the thinker's maturity, and as he grows in knowledge and skill and in his ability to sense relationships, his need for higher standards also will grow.

Not all children in the elementary school are equally able. They do not all have the same ability, and as we try to provide opportunities for them to add depth to their learning, it appears that we may attempt the impossible.

Ambitious goals that we set up will be attained fully by some, to a modest degree by many, and to a limited degree by others. Nevertheless, it is sound policy to encourage all children to think as hard as they can and to understand as much as they can. It is better to see through a glass darkly than not to see at all. The opportunity to attain adequate understanding and to add depth to their learning should be open to all children.

The unquestionable evidence of differences in learning ability must be faced squarely by those of us responsible for instruction in arithmetic. This need not lead to a pessimistic or defeatist point of view, but to the establishment of attainable goals and the devising of more effective methods for their accomplishment.

Depth Learning In Arithmetic—What Is It?*

Charlotte W. Junge

Today, there appears to be common agreement among teachers of arithmetic that there should be little, if any, differentiation of topics in the elementary school to meet individual needs. The differentiation, when made, should be in terms of depth and scope of learning. As Roland Smith states, "....the amount of concrete background in any topic can be varied, and each child can be expected to do work only up to his own capacity to learn."[2]

Providing for differentiation in scope and in depth is made easier by a method of teaching which leads children to think as a mathematician would think. It enables them to solve problems that are not exactly like those solved in the classroom or in the textbook. Furthermore, it enables them to deal precisely with precise ideas, and to work with symbols as an aid to thought, full of meaning and conveying ideas which can be understood.

SELECTED EXPERIENCES:

Children are particularly delighted with learning carried on in this way. Recently I watched children in a second grade, who had developed some facility with basic addition and subtraction facts, work through a series of questions in which they were solving for the "missing number." For example:

$5 + \square = 7$ $\qquad$ $\triangle + 9 = 10$

$2 + \square = 8$ $\qquad$ $\triangle + 6 = 13$

Each child was encouraged to tell "how he thought" in finding the missing numbers. Then the children were asked to work independently to find pairs of numbers which would make these equations true:

$\triangle + \square = 7$ $\qquad$ $\square + \triangle = 5$

Class evaluation revealed that several different number pairs could be used in each example. Some of the children who experienced no difficulty here were encouraged to go on to the slightly more difficult situation of:

A	B
$6 + 3 + \square = 10$	$6 + 2 + \square = 10$
$4 + \triangle + 3 = 9$	$4 + 2 + \triangle = 9$
$\square + 2 + 4 = 8$	$1 + \triangle + 4 = 8$

The children were encouraged to create similar exercises with which to challenge their classmates.

At a fifth or sixth grade level, I watched children, who have had opportunities to do this type of work, extend their thinking with exercises in which they decided whether a number sentence was true or false depending on the numbers used. For example:

Depth Learning In Arithmetic—What Is It?*

Charlotte W. Junge

A. $3 + 5 = 8$ (true)
   $2 + 7 = 8$ (false)
B. Is the number sentence, $A + 4 = 9$, true?
   .if $A = 4$
   .if $A = 5$
   .if $A = 6$
C. Is this sentence, $A + B + 6 = 12$, true or false?
   .if $A = 3$ and $B = 4$
   .if $A = 2$ and $B = 4$
   .if $A = 2$ and $B = 3$
   .if $A = 2\frac{1}{2}$ and $B = 3\frac{1}{2}$
   .if $A = 5.2$ and $B = .8$

The children were encouraged to find other pairs of numbers which made the sentence true and to share their thinking with the whole class.

Also, I observed a sixth grade teacher working with a group of children of varying abilities. The children worked as a group to solve the following:

.If $T = 4$; $R = 6$, and $V = 5$, find the number which will make each of the following number sentences true.
.$R + T + V = ?$
.$T \times V \times R = ?$
.$(R - T) + V = ?$
.$T + R \div V = ?$

After solving several exercises like these, the children then developed other similar exercises to share with the class.

As children develop understandings and competence with the basic mulitplication facts, they may be given opportunities to stretch their thinking on exercises like this:

$$\begin{aligned} 2 \times 9 &= (2 \times 4) + (2 \times 5) \\ 5 \times 7 &= (5 \times 2) + (5 \times 5) \\ &= (5 \times 3) + (5 \times 4) \\ &= (5 \times 1) + (5 \times 6) \end{aligned}$$

Can you complete the following by finding other ways to express the meaning of 4×6, 5×8, 6×7?

$4 \times 6 =$ _____ $5 \times 8 =$ _____ $6 \times 7 =$ _____

After instruction on how to multiply without pencil and paper pupils in one fourth grade were using six different ways of arriving at the product of 12×25:

Depth Learning In Arithmetic—What Is It?*

Charlotte W. Junge

1. $10 \times 25 = 250$
   $2 \times 25 = 50$
   $12 \times 25 = 300$

2. $4 \times 25 = 100$
   $4 \times 25 = 100$
   $4 \times 25 = 100$
   So $12 \times 25 = 300$

3. $12 = 4 \times 3$
   $3 \times 25 = 75$
   $4 \times 75 = 300$
   So $12 \times 25 = 300$

4. $12 = 3 \times 4$
   $4 \times 25 = 100$
   $3 \times 100 = 300$
   So $12 \times 25 = 300$

5. 25×12
   50
   250
   300

6. $12 = 2 \times 6$
   $6 \times 25 = 150$
   $2 \times 150 = 300$
   So $12 \times 25 = 300$

Children whose written work indicates that they require little additional practice in multiplication and division profit from exercises which develop relational thinking. For example:

Tell how you think to get the answer

1. $32 \times 16 = 512$ (a) 32×8 (b) 32×32 (c) 64×16 (d) 16×32

2. $18\overline{)216} = 12$ (a) $12\overline{)216}$ (b) $36\overline{)216}$ (c) $18\overline{)432}$ (d) $18\overline{)?}$

Having children tell how they think when finding trial quotients in division is not only helpful to individual children, but it helps all children in the class deepen their thinking. Recently, in a fifth grade, I watched children share their solutions to the following examples.

$29\overline{)89}$ Bob: The answer can't be as much as 4, since 29 is almost 30 and $4 \times 30 = 120$. The answer will be a little more than 3.

$51\overline{)426}$ Sue: There are 2 50's in 100. In 400 there will be at least 4 times as many. I'll try 8 as a quotient figure.

$45\overline{)280}$ Ann: There are 2 45's in 100. In 300 there will be about 6. I'll try 6 as a quotient figure.

Occasionally current happenings can be used to bring depth into children's learning. When Pioneer IV was successfully placed in orbit, a fifth grade child related to his classmates that the velocity of the rocket was a little less than 25,000 mph at time of launching, that at X + 1 the speed was about 11,800 mph, and at X + 2 the speed was reduced to about 7,800 mph. Class discussion clarified for the children the difference between velocity and speed. The children noted that the rocket's speed at the end of the first hour was about half that at time of launching, and that at the end of the second hour the speed was reduced to about one-third the original speed. A speed of 25,000 mph had little meaning for the children until a little computation indicated that this was about 400 miles per minute. Knowing that it is about 400 miles across Michigan from north to south, the children began to gain an appreciation of the rock-

Depth Learning In Arithmetic—What Is It?*

Charlotte W. Junge

et's speed when they realized it would flash across Michigan in one minute.

Another computation indicated that the rocket traveled more than 6 miles a second and, as one child said, "Oh, we would travel from Detroit to Ann Arbor in about 7 sec!"

When children have gained an understanding of the decimal number system, scientific notation may be introduced in grade six as a shorter and faster way of working with large numbers. For example, the children have learned that in the decimal number system:

1. $10 = 10 \times 1 \longrightarrow 10 = 10^1$ (10 used as a factor once)
   $100 = 10 \times 10 \longrightarrow 100 = 10^2$ (10 used as a factor twice)
   $1{,}000 = 10 \times 10 \times 10 \longrightarrow 1{,}000 = 10^3$ (10 used as a factor 3 times)
   $10{,}000 = 10 \times 10 \times 10 \times 10 \longrightarrow 10{,}000 = 10^4$ (10 used as a factor 4 times)

They can extend this to include:

2. $10 \times 100 =$? or $10 \times 10^2 =$?

3. $100 \times 1000 =$? or $10^2 \times 10^3 =$?

From exercises like these children can be led to discover that, when mulitplying powers of the same base, they can add the exponents.

1. $10^3 \times 10^2 = 10 \times 10 \times 10 \times 10 \times 10 \times 10^5 = 100{,}000$

2. $10^2 \times 10^2 = 10^{2+2} = 10^4 = 10{,}000$

3. $2^3 \times 2^1 = 2^4 = 2 \times 2 \times 2 \times 2 = 16$

If the teacher thinks it advisable, the use of a dot to indicate multiplication may be introduced at this time, too, and children may be encouraged to write 2×5 as $2 \cdot 5$; 6×7 as $6 \cdot 7$, etc.

In helping children extend their understanding of the use of formulas to express relationships, a teacher of eleven-year-olds placed the following number series on the chalkboard and asked if they could find a quick way to find the sum of all numbers from 1 to 101 without adding:

a. $1\ 2\ \underline{3}\ 4\ \underline{5} = 15$

b. $1\ 2\ 3\ 4\ \underline{5}\ 6\ 7\ 8\ \underline{9} = 45$

c. $1\ 2\ 3\ 4\ 5\ \underline{6}\ 7\ 8\ 9\ 10\ \underline{11} = ?$

d. $1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13 = ?$

After a little study the children said that the sum in each series was equal to the middle number times the last number in the series. This they expressed as "Sum = M x L." Applying this formula to the original question, they found the sum of all numbers from 1 to 101 to be 51 x 101 or 5151.

Depth Learning In Arithmetic—What Is It?*

Charlotte W. Junge

Then the teacher called attention to the fact that the number series with which they had been working all ended with odd numbers. She asked, "Will the formula work with a number series ending with even numbers?"

1 2 3 4 5 6 = 21

1 2 3 4 = 10

1 2 3 4 5 6 7 8 9 10 = 55

It was necessary to develop new formula for these series and the children developed three different ones:

1. (First + Last) x ($\frac{1}{2}$ of the Last)
   (F + L) x ($\frac{1}{2}$ x L

2. (Last + 1) x ($\frac{1}{2}$ of the Last)
   (L + 1) x ($\frac{1}{2}$ x L)

3. (Last x Middle) or L x M.

In this latter case the children worked with fractions and in the series "1 2 3 4 5 6" used the formula L x M = 6 x $3\frac{1}{2}$ = 21.

Recently I observed a fourth grade group working with number sequences. They had started with very simple sequences such as:

2, 4, 6, 8
7, 9, 13, 21

and were asked to write the next four numbers, then they advanced to irregular patterns such as 12, 14, 15, 17 and continued to write "the next four numbers." I watched a small girl place the following sequence on the blackboard and ask her classmate to complete it :

32 30 26 20 22 __ __ __ __

It was the same class who, at an earlier date, had worked successfully with numbers to the base of 7!

IN CONCLUSION:

These are but a few illustrations of learning experiences used to add depth in meaning for children in arithmetic. The rule which we follow in achieving this kind of learning is a simple one:

Broaden the contest and lift the skill to the conceptual level, by teaching so that children understand relationships, extend them to solutions of new problems, and have time to think, to question, to wonder.

Depth Learning In Arithmetic—What Is It?*

Charlotte W. Junge

To do this the teacher must be concerned with the development of the individual student. Each student builds his own ideas, and it is perfectly clear that in so doing his attempts will be attended by trial and error. It is equally clear that there is no substitute for the teacher either in stimulating the child's thinking or correcting and perfecting the ideas he forms. No administrative device, no amount of instructional equipment, important as it is, can take the teacher's place. Because of the complex nature of learning in mathematics, it is crucially important that teachers have adequate preparation. We, too, must add depth to our learning!

FOOTNOTES

[1]Morris R. Cohen, Reason and Nature (New York: Harcourt Brace & Company, 1931,) page 17.

[2]Roland Smith, "Provisions for Individual Differences," Twenty-first Yearbook (Washington, D.C.: NCTM, 1953), page 273.

* From The Arithmetic Teacher, No. 7 (November 1960), 341-346.

Now that large-scale compensatory education for culturally disadvantaged children is a fact, a new challenge arises for educators: To find the way to get right on target in supplying the needed-for-learning ingredients missing in a culturally deprived home.

It is not easy to decide just what kind of preschool program will work best with disadvantaged youngsters. Some argue, for example, that a good traditional nursery school program is enough. Others would add to the traditional program experiences that match up with home deficits-trips to the zoo, opportunities to care for animals, and other experiences typical of those that culturally privileged children enjoy with their families. Head Start programs, for the most part, have been based on this enrichment principle.

TOO MUCH TO CHANCE?

There are still others, however, who argue that the traditional nursery school, even with enrichment experiences, leaves too much to chance. They favor planned intervention by means of a structured curriculum geared to the special cognitive needs of the disadvantaged. Perhaps all the compensatory programs that are university-research centered fall into this category, but the degree and kind of intervention vary considerably.

Some "interventionists," for example, support teaching the three Rs in preschool. Others emphasize perceptual training. Still another approach is the one developed by Jean Piaget, the Swiss psychologist, who puts the accent on developing logical thought processes.

Piaget would consider both the three-R and perceptual approaches too limited--the first because it offers little in the way of generalized how-to-learn skills and the latter because it teaches a child to depend on sensory data that are often misleading. Instead, a Piaget-type program would involve all the features of a good nursery school, plus:

- Activities to facilitate development of mental structures from the simple to the complex--combining objects, taking groups apart, ordering objects, setting objects in one-to-one correspondence, considering two properties of an object at a time, and simple conservation tasks.

- Transactional rather than perceptual experiences. A young child thinks he has more bubble gum when he pulls it out in a long string than when the gum is in a ball. He cannot learn that a change in one dimension is compensated for by a change in another until he learns to operate on data and not merely perceive them.

- Concentrated small-group instruction in the type of structure-inducing activities that the advantaged child carries on in the course of daily living.

- Maximum opportunity for generalization. Teachers must look for incidental opportunities throughout the day to give practice in combining, ordering, conserving, etc. Thus, training in one-to-one correspondence, for example, is reinforced as a child sets the table at snack time, putting out one paper cup and napkin for each person at the table.

- Language emphasis through-out the day and especially during periods devoted to cognitive activities. Children must be encouraged to put into words what is happening.

- Home-school teamwork. With the cooperation of a child's family, intellectual stimulation can be made to go beyond the short nursery school day. The family needs to know what the school is doing, and why, and what they can do to help.

* From: Grade Teacher, Oct., 1967, pp. 123-127, By Celia B. Lavatelli.

Here are some examples of how the principles outlined above take shape in actual classroom practice:

- Orientation Exercises:

Show a circle cutout. Have pupils draw circles freehand. If after checking their work against the model, it is found to be way off base, have them trace cardboard circles with their fingers. When the circle is mastered (it should look more round than rectangular), follow the same procedure with the square and triangle.

Have the children color a circle red. Introduce additional colors, encouraging them to use vocabulary, as Q: "What are you making?" A: "A yellow circle."

Introduce sizes ("big," "little") using geometric shapes as examples.

Invent Games:

Invent games to reinforce responses. For example: Give each child a Bingo-type card with two rows of geometric shapes varied in size and color. Ask questions like: "Who has a large red circle?"

- Learning that an object can have more than one property through one-for-one matching:

String geometrically shaped beads in a simple order, such as square, circle,square, circle etc. Have each child copy the model and check his work by one-to-one correspondence.

Vary the arrangement--for example, square, square, circle, square, square, etc. Gradually introduce more complexity into the patterns.

Introduce color as a variable and then size as a variable. In case of error, demonstrate again for the child the technique of one-to-one correspondence.

- Combining objects to form classes:

Place pictures of various objects at the center of a table. Assign each child a picture. He is to search for others that are like it in some way.

- Multiplicative classification:

Each child is given geometric shapes--small red squares and large squares in either red or green. Questions to ask: Do all of the small things belong in this box with the reds? This is a box for squares; do the greens go in this box? Do the reds go in the box for squares? This is a box for small shapes; do the greens belong in it?

- Complementary classes:

In the center of a table, place pictures of fruits (red apples and strawberries; yellow bananas and grapefruit; orange oranges and pumpkins) and pictures of miscellaneous non-living objects that are red, yellow and orange. Supply each child with three shallow boxes. Then:
(1) Ask the children to collect pictures of fruit in one box and non-fruit item in another.
(2) Have them put all yellow fruits one box, red ones in another, and orange fruits in a third.
(3) Have them put all the yellow objects in one box, red in another, and orange in the third.

-"Whole-part" relationships:

Have the children set up rows of 10 geometric shapes, varied in shape and color. Ask questions like: Are all of the reds squares? Are all of the squares red? Are all of the blues triangles?

Count the boys and girls in the class. Count all the children in the class. Ask questions like: Are there more boys or more girls: Are there more children or more boys?

- Intersection of classes:

The children put green blocks in one ring. Large blocks in varying colors are enclosed in a second ring. The teacher slides one ring partly over the other. Ask the children what goes in the intersection.

- Finding all possible combinations of two variables:

Provide each child with two paper dolls. Provide him also with pictures of four red and four yellow sweaters and four blue and four green skirts (or trousers). He must dress the dolls in as many outfits as possible, not repeating any combination.

When I was in sixth grade the teacher used to propose special math problems and give us a half hour to solve them. Any student who got the right answers in that time was given a special high grade. I never got any of the problems right in school and kicked myself for being so stupid.

I used to take the problems home with me, hide in my room and think about them. At home, alone, able to listen to music and walk around I could solve the problems. In school with all the other students around me, with the silence in the room and the pressure of grades and time I couldn't think about math. All I thought about was not getting the right answer and being thought stupid by the teacher and the other kids.

When I taught the sixth grade there were many fights in the class. Students often defied me, but more often they fought with each other. I had to resolve the fights as quickly and arbitrarily as possible. There was no space for working out the problems behind the fights or in which to let combatants rest and talk to each other.

I remember a particularly bad fight between two boys who had liked each other very much up until the moment the battle broke out in the classroom. I wanted to go and talk with the boys but that was impossible. There was no one to watch my class, and even if there had been, the only place we could have talked was in the hall which was full of movement and very unprivate.

During the time I taught in New York City the union won preparation periods for elementary school teachers. We had an hour a day to prepare lessons and do research on curriculum. The problem was that the only place to do it was the teachers' lunchroom which was filled with people drinking coffee and gossiping or just resting and looking for a conversation.

Just this year I have talked with a number of students about how to make the school environment more comfortable and welcoming. Almost without exception the students wanted spaces which were private, in which they could be alone, or alone with a few friends, a place where they could talk and work and think. One of their main complaints about school was that there was no provision for the privacy and personal lives of students. In some schools the doors are removed from the stalls in the student toilets.

There are certain things best done in private--among them thinking, creating, solving problems, resolving conflicts and resting. Schools don't provide private places for the students or teachers to think or work alone or develop environments that fit their personal style of learning and teaching. They don't have places where people can come together in small groups and work free from public scrutiny.

The classroom is a particularly public place. The space is open with no private corners or separate divisions for people to be alone in. Yet it is the classroom where we expect and often demand that students think and solve problems and create poems and stories, and deal with their conflicts with other students.

Let us look at these different activities separately. How do people usually go about thinking and solving problems, creating works of art and solving conflicts outside the classroom?

Thinking is a complex activity which no one has been able to quantify or describe fully. It is clear, however, that different people find themselves able to think best in different circumstances. Some people think best with music on, others need complete silence. Some people do their best thinking while walking, others while sitting in an easy chair, or lying down, or sitting in a public square watching people they don't know pass by.

* From: Grade Teacher, Jan., 1970, pp. 10-12, By Herbert Kohl

Problems are solved in many different ways. Some people dream up answers, others need a pen in their hand or another person to push them into thinking of a solution.

Making a work of art is an even more eccentric activity. Artists and writers have special conditions and environments under which they work. Some can work with other people present, others need to be completely alone. I cannot write with anyone else around. I also have to write on a certain size paper and with a special pen. This is not just my madness. We all seek the most comfortable environments in which to work. Robert Creely, the poet, expressed what I am getting at in a speech he gave at a poetry conference in Vancouver, Canada, some time ago. He said:

"What I'm trying to say is don't start thinking of writing as some particular activity leading to some particular effect for some particular purpose. It is just as relevant what size paper you use, as whether or not you think you are writing a sonnet. In fact, it's more relevant. And this aspect of your activity ought to be, you ought to be aware of it. . . In other words if you want to write with a paper like this, please DO! If you find yourself stuck with habits of articulation try doing something else, try shifting the physical context. . ."

Solving conflicts is no less personal than creating a work of art. I do not feel easy dealing with someone's problem in public. As a teacher I was always embarrassed by the need to deal with problems of two or three students with the other students present.

It is important to build private spaces into existing classrooms. Right now I am experimenting with the possibility of creating private corners and little rooms within existing rooms. Fabric is magical. A few muslin dividers hanging from the ceiling can create private spaces. Clothing closets can be turned into private rooms. Rugs on the floor can mark private spaces. Portable Japanese walls or ingenious room dividers can be used.

In short, the classroom can be changed into a complex environment accommodating private as well as public experience. There is no simple formula to achieve this transformation. It is up to the ingenuity of individual teachers. We have to take the environments that we are given as teachers and change them into fine places for living and learning.